Decidable fragments of first-order logic

For the moment, let L be a vocabulary of first-order logic that for every arity contains countably many relations symbols of this arity.

Definition

A sentence over L is *valid* if it holds in all L-structures.
A sentence over L is *satisfiable* if it holds in some L-structure.

Theorem

The set of of valid sentences over L of first-order logic is recursively enumerable, but not decidable.

That the set of valid L-sentences is not decidable means that there is no effective procedure that on any input eventually terminates and correctly decides whether the input is valid or not.

A sentence is valid if and only if its negation is not satisfiable, thus satisfiability of an L-sentence is not decidable, either.

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Definition

A set of sentences has a *decidable satisfiability problem* if there is an effective procedure that correctly decides for any given sentence from this set whether the sentence is satisfiable (the procedure might behave arbitrarily on inputs that are not in the given set).

The monograph *The Classical Decision Problem* by Börger, Grädel, and Gurevich contains an extensive survey on results that assert that certain fragements of first order-logic have a decidable satisfiability problem or not.

Here fragments of first-order logic are distinguish according to

- whether the equality symbol can be used,
- how many relation and function symbols of the different arities can be used,
- what types of quantifier prefixes are allowed (where sentences are assumed to be in prenex form).

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Theorem

Let L be a vocabulary that contains only a single relation symbol, which is binary.
Then for each of the following types of quantifier prefix, the class of sentences over L without equality that are in prenex form with and have a quantifier prefix of this type has an undecidable satisfiability problem.

(a) $\forall^*\exists$, (b) $\forall\exists\forall^*$, (c) $\forall\exists\forall\exists^*$, (d) $\forall^3\exists^*$,
(e) $\forall\exists^*\forall$, (f) $\exists^*\forall\exists\forall$, (g) $\exists^*\forall^3\exists$.

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Definition

A $\forall^2\exists^*$-*sentence* is a sentence of the form

$$\forall x_1 \forall x_2 \exists x_3 \ldots \exists x_{l+2} \psi$$

where $l \geq 0$ and ψ is quantifier-free.

A formula is *relational* if it does not contain constant or function symbols (but just relation symbols).

A formula *without equality* is a formula that does not contain the equality symbol.

Theorem

The set of relational $\forall^2\exists^*$-sentences without equality over any fixed vocabulary has a decidable satisfiability problem.

The theorem can be extended to $\exists^*\forall^2\exists^*$-sentences.

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The theorem is demonstrated by showing that the set of sentences under consideration has the finite model property.

Definition

A set of sentences has the *finite model property* if every satisfiable sentence in the set has a finite model.

Proposition

Every set of sentences with the finite model property has a decidable satisfiability problem.

Proof of the proposition.

In a nutshell, the idea for obtaining a decision procedure asserted by the proposition is the following: for a given sentence φ search in parallel for a finite model of φ and for a proof of $\neg\varphi$.

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Proof of the proposition (continued).

In order to decide the satisfiability of an input φ, run the following two processes in parallel.

> Process 1 seaches for a finite model of φ by a possibly infinite exhaustive search,
>
> Process 2 tries to verify that $\neg\varphi$ is valid by enumerating all the valid sentences.

Consider an input φ that is indeed in the given set.

In case φ is satisfiable, then it has a finite model and the first process eventually halts whereas the second one runs forever.

In case φ is not satisfiable, then $\neg\varphi$ is valid and consequently the second process eventually halts, whereas the first one runs forever.

So we can simply wait until one of the processes halts. □

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Proposition

The set of relational $\forall^2\exists^*$-sentences without equality has the finite model property.

Definition

For a relational formula φ, let L_φ be the finite vocabulary that contains exactly the relation symbols occurring in φ.

Remark

A relational formula φ is true in some structure if and only if the formula is true in an L_φ-structure, and a similar equivalence holds for finite structures and finite L_φ-structures.

For a proof, observe that any structure in which φ is true can be reduced to an L_φ-structure in which φ is true.

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Definition

Let L be a vocabulary that contains only relation symbols and let $\mathfrak{A}$ be an L-structure with universe A.

A *k-table* over L is an L-structure with universe $\{1,\dots,k\}$.

For mutually distinct elements $a_1,\dots,a_k$ in A, *the k-table induced by $a_1,\dots,a_k$ in $\mathfrak{A}$* is the unique k-table that is isomorphic via the mapping $i \mapsto a_i$ to the substructure of $\mathfrak{A}$ induced by the a_i.

A k-table is *realized in* $\mathfrak{A}$ if it is induced by some k-tuple over A.

An element of A is a *king* if it induces a 1-table that differs from all 1-tables induced by other elements of A.

Accordingly, the structure $\mathfrak{A}$ is a *structure without kings* if any 1-table that is realized in $\mathfrak{A}$ at all is induced by at least two distinct elements of A.

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Lemma

Every satisfiable relational sentence φ without equality is true in an L_φ-structure without kings.

Sketch of proof.

Fix any satisfiable relational sentence φ without equality and suppose that φ is true in some L_φ-structure $\mathfrak{A}$ with universe A.

For any set I, let $\mathfrak{A} \times I$ be the unique L_φ-structure with universe equal to $A \times I$ such that for every k-ary relation symbol in L_φ, all $a_1, \ldots, a_k$ in A, and all $i_1, \ldots, i_k$ in I,

$$R((a_1, i_1), \ldots, (a_k, i_k)) \text{ is true in } \mathfrak{A} \times I \iff R(a_1, \ldots, a_k) \text{ is true in } \mathfrak{A} .$$

If I has at least two elements, then in $\mathfrak{A} \times I$ there is no king.

Induction on the structure of formulas shows that in $\mathfrak{A}$ and in $\mathfrak{A} \times I$ the same sentences over L_φ are true. □

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Lemma

Let φ be a relational $\forall^2\exists^*$-sentence. If φ is true in an L_φ-structure without kings, then φ has a finite model.

Proof.

Fix any L_φ-structure $\mathfrak{A}$ without kings in which φ is true.

We give a randomized construction that for any given n yields an L_φ-structure $\widehat{B}_n$ with universe $B_n = \{1, \ldots, n\}$.

For sufficiently large n, the probability that the construction results in a model of φ will be strictly larger than 0, hence there is a finite model of φ.

Applications in Logic

Proof (continued).

For $r = 1, 2$, let T_r be the set of all r-tables that are realized in $\mathfrak{A}$. Both sets are finite because L_φ is finite.

The random L_φ-structure $\widehat{\mathfrak{B}}_n$ with universe $B_n = \{1, \ldots, n\}$ is obtained as follows.

(i) To each b in B_n, assign a 1-table $\mathfrak{T}_b$ that is chosen uniformly at random from T_1.

(ii) To each subset $\{b_1, b_2\}$ of B_n where $b_1 < b_2$, assign a 2-table $\mathfrak{T}_{\{b_1,b_2\}}$ that is chosen uniformly at random from the set of all 2-tables in T_2 where the 1-tables induced by the elements 1 and 2 are $\mathfrak{T}_{b_1}$ and $\mathfrak{T}_{b_2}$, respectively.

(iii) For any relation symbol R of arity k in L_φ and all $b_1, \ldots, b_k$ in B_n such that the truth value of $R(b_1, \ldots, b_k)$ has not already been determined during steps (i) and (ii), decide the truth value of $R(b_1, \ldots, b_k)$ by tossing a fair coin.

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Proof (continued).

It remains to show that for n sufficiently large, φ is true in $\widehat{\mathfrak{B}}_n$ with nonzero probability.

Assume that φ can be written for quantifier-free ψ in the form

$$\varphi \;\equiv\; \forall x_1 \forall x_2 \exists x_3 \ldots \exists x_{l+2} \psi .$$

Then φ is true in any given L_φ-structure if for every nonempty subset $\{i_1, i_2\}$ of this structure's universe there are $i_3, \ldots, i_{l+2}$ in the universe such that $\psi[i_1, \ldots, i_{l+2}]$ is true.

Proof (continued).

First, consider an arbitrary subset $\{b_1, b_2\}$ of B_n of size 2.

Fix some 2-table $\mathfrak{T}$ in T_2 and assume that $\mathfrak{T}$ has been assigned to $\{b_1, b_2\}$ during Step ii of the construction of $\widehat{\mathfrak{B}}_n$.

Recall that $\mathfrak{T}$ is realized in the structure $\mathfrak{A}$ and that $\mathfrak{A}$ satisfies φ.

Thus we can extend $\mathfrak{T}$ to an $(l+2)$-table $\mathfrak{T}^{\mathrm{ext}}$ where

- $\mathfrak{T}^{\mathrm{ext}}$ is realized in $\mathfrak{A}$,
- $\psi[1, 2, i_3, \ldots, i_{l+2}]$ is true in $\mathfrak{T}^{\mathrm{ext}}$ for not necessarily pairwise distinct numbers $i_3, \ldots, i_{l+2}$ in $\{1, \ldots, l+2\}$.

Then for any list of numbers $b_3, \ldots, b_{l+2}$ in B_n that are mutually distinct and differ from b_1 and b_2, the table induced by $b_1, \ldots, b_{l+2}$ in $\widehat{\mathfrak{B}}_n$ will be equal to $\mathfrak{T}^{\mathrm{ext}}$ with probability $\varepsilon(\mathfrak{T}) > 0$, where $\varepsilon(\mathfrak{T})$ depends neither on n nor on the b_j.

Proof (continued).

We can argue similarly for any subset $\{b\}$ of B_n of size 1.

Fix some 1-table $\mathfrak{T}$ in T_1 and assume that $\mathfrak{T}$ has been assigned to $\{b\}$ during Step i of the construction of $\widehat{\mathfrak{B}}_n$.

Then we can extend $\mathfrak{T}$ to an $(l+1)$-table $\mathfrak{T}^{\mathrm{ext}}$ where

- $\mathfrak{T}^{\mathrm{ext}}$ is realized in $\mathfrak{A}$,
- $\psi[1, 1, i_3, \ldots, i_{l+2}]$ is true in $\mathfrak{T}^{\mathrm{ext}}$ for not necessarily pairwise distinct numbers $i_3, \ldots, i_{l+2}$ in $\{1, \ldots, l+2\}$.

Then for any list of numbers $b_3, \ldots, b_{l+2}$ in B_n that are mutually distinct and differ from b, the table induced by $b, b_3, \ldots, b_{l+2}$ in $\widehat{\mathfrak{B}}_n$ will be equal to $\mathfrak{T}^{\mathrm{ext}}$ with probability $\varepsilon(\mathfrak{T}) > 0$, where $\varepsilon(\mathfrak{T})$ depends neither on n nor on the b_j.

Proof (continued).

Fix any subset $\{b_1, b_2\}$ of B_n of size $r \in \{1, 2\}$ and recall that some r-table $\mathfrak{T}_{\{b_1,b_2\}}$ from T_r is assiged to this subset.

For any subset $\{b_3, \ldots, b_{l+2}\}$ of pairwise distinct elements of B_n that differ from b_1 and b_2, consider the event that the table induced by $b_1, \ldots, b_{l+2}$ is equal to $\mathfrak{T}^{\mathrm{ext}}_{\{b_1,b_2\}}$; then

- each such event has probability at least ε,
- for any list of such subsets that are mutually disjoint, the corresponding events are mutually independent.

For large n, there are at least $\frac{n}{2l} \leq \lfloor (n-2)/l \rfloor$ such subsets that are mutually disjoint, hence the probability that $\mathfrak{T}^{\mathrm{ext}}_{\{b_1,b_2\}}$ is not induced by any of these subsets can be bounded from above by

$$(1-\varepsilon)^{\frac{n}{2l}} = [(1-\varepsilon)^{\frac{1}{2l}}]^n.$$

Proof (continued).

If we let

$$\delta \;=\; (1-\varepsilon)^{\frac{1}{2l}} < 1\,.$$

then for any subset $\{b_1, b_2\}$ of B_n , the probability that in $\widehat{\mathfrak{B}}_n$ there are no elements $b_3, \ldots, b_{l+2}$ as required is at most δ^n.

By summing up over the less than n^2 nonempty sets $\{b_1, b_2\}$ in $\widehat{\mathfrak{B}}_n$, the overall "error probability" can be bounded by

$$n^2\delta^n \quad \stackrel{n\to\infty}{\longrightarrow} \quad 0\,.$$

Consequently, the probability that the sentecnce φ is true in $\widehat{\mathfrak{B}}_n$ tends to 1 when n increases. □