The Lovasz Local Lemma

Avoiding certain events simultaneously

Suppose that $E_1, \dots, E_n$ are events that correspond to errors or other outcomes that we want to avoid.

In what follows, we investigate conditions on the underlying probability distribution that imply that the events E_i can be avoided simultaneously, in the sense of implying nonzero probability for the event

$$G = \bigcap_{i=1,\dots,n} \bar{E}_i \; .$$

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Sum of probabilities

If the sum $\mathrm{Prob}[E_1] + \cdot + \mathrm{Prob}[E_n]$ is strictly less than 1, then the event G has nonzero probability.

A necessary condition for G having nonzero probability is that all E_i have probability strictly smaller than 1, where the latter condition is in fact equivalent in case of mutual independence.

Independence

If the events $E_1, \dots, E_n$ are mutually independent, then G has nonzero probability if and only if for all i holds $\mathrm{Prob}[E_i] < 1$.

For a proof, it suffices to observe that by independence we have

$$\mathrm{Prob}[G] = \mathrm{Prob}\left[\bigcap_{i=1}^{n} \bar{E}_i\right] = \prod_{i=1}^{n} \mathrm{Prob}[\bar{E}_i] = \prod_{i=1}^{n} (1 - \mathrm{Prob}[E_i]) \; .$$

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Definition

Let $X_1, \dots, X_n$ be random variables with ranges $A_1, \dots, A_n$, respectively.
The random variable X_i is *mutually independent* of the random variables $X_{i_1}, \dots, X_{i_t}$ if for all $a_i \in A_i$ and all $a_{i_j} \in A_{i_j}$

$$\mathrm{Prob}[X_i = a_i | X_{i_1} = a_{i_1} \& \cdots \& X_{i_t} = a_{i_t}] = \mathrm{Prob}[X_i = a_i] \; .$$

In the situation of the last definition, we say that the random variable X_i depends among $X_1, \dots, X_n$ only on the set of all events X_j where j is in the set $D = \{1, \dots, n\} \setminus \{i_1, \dots, i_t\}$.

The notation above is extended to events $E_1, \dots, E_n$ by considering the corresponding indicator variables, i.e., E_i is mutually independent of $E_{i_1}, \dots, E_{i_t}$, if the indicator variable of E_i is mutually independent of the indicator variables of the $E_{i_1}, \dots, E_{i_t}$.

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Theorem (Lovasz Local Lemma)

Let $E_1, \dots, E_n$ be events and let $D_1, \dots, D_n$ be subsets of $\{1, \dots, n\}$ such that for $i = 1, \dots, n$, the event E_i depends among $E_1, \dots, E_n$ only on the set of all events E_j where $j \in D_i$. Furthermore, assume that there are real numbers $x_1, \dots, x_n$ in the interval $[0, 1]$ where

$$\mathrm{Prob}[E_i] \le x_i \prod_{j \in D_i} (1 - x_j) \; .$$

Then it holds for the event $G = \bigcap_{i=1,\dots,n} \bar{E}_i$ that

$$\mathrm{Prob}[G] \ge \prod_{i=1,\dots,n} (1 - x_i) \; .$$

(The proof of the Local Lemma is still missing.)

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Corollary (Basic form of the Lovasz Local Lemma)

Let $E_1, \ldots, E_n$ be events where $\mathrm{Prob}[E_i] \leq p$ for some real $p < 1$. Furthermore, assume that there is a constant d where

$$\mathrm{e}(d+1)p \leq 1$$

(for Euler's number $\mathrm{e} = 2.71\ldots$) and there are sets $D_1, \ldots, D_n$ of size at most d such that for $i = 1, \ldots, n$, the event E_i depends among $E_1, \ldots, E_n$ only on the set of all events E_j where $j \in D_i$.
Then it holds for the event $G = \bigcap_{i=1,\ldots,n} \bar{E}_i$ that

$$\mathrm{Prob}[G] \geq 0 .$$

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Proof of the corollary from the local lemma.

For $d = 0$, the events E_i are mutually independent, hence the conclusion of the corollary is immediate due to $\mathrm{Prob}[E_i] < 1$.

So assume $d > 0$ and let $x_i = 1/(d+1)$.
Then the assumption of the Local Lemma is satisfied because it holds for $i = 1, \ldots, n$ that

$$x_i \prod_{j \in D_i} (1 - x_j) \geq \frac{1}{d+1} \left(1 - \frac{1}{d+1}\right)^d \geq \frac{1}{d+1} \cdot \frac{1}{\mathrm{e}} \geq p \geq \mathrm{Prob}[E_i] ,$$

where the second inequality holds because $(1 - \frac{1}{d})^d$ converges nonincreasingly to $1/\mathrm{e}$. The Local Lemma then yields

$$\mathrm{Prob}[G] \geq \prod_{i=1,\ldots,n} (1 - x_i) = \left(1 - \frac{1}{d+1}\right)^n > 0.$$

□

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The proof of the following proposition provides an example for an application of the Lovasz Local Lemma.

Recall that a Boolean formula φ is in k-conjunctive normal form (k-CNF) if φ is a conjunction of clauses, where each clause contains at most k literals.

Note that for the following proposition, we use a notion of k-CNF where it is required in addition that every clause contains exactly k literals, where the variables of these literals are pairwise distinct.

Proposition

Let φ be a Boolean formula in k-CNF such that every variable occurs in at most $2^{k - \log k - 2}$ clauses of φ. Then φ is satisfiable.

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Proof.

Consider the random experiment where the variables in φ are assigned truth values by independent tosses of a fair coin.

Let E_i be the event that the ith clause of φ is not satisfied.

By construction, we have $p = \mathrm{Prob}[E_i] = 1/2^k$.

Each E_i is mutually independent of the set of all E_j such that the ith and the jth clause of φ do not have a variable in common.

Thus by assumption on φ, any event E_i depends among the events E_j only on a set of at most $d = k2^{k - \log k - 2}$ events, where

$$\mathrm{e}dp = \mathrm{e}k2^{k - \log k - 2}2^{-k} < 1 .$$

Hence the basic form of the Lovasz Local Lemma applies and φ will be satisfied with nonzero probability. □