Sum-free subsets

Definition

A set Z of integers is *sum-free* if for all x, y, z in Z holds $x + y \neq z$.

Theorem

Every nonempty finite set Z of integers contains a sum-free subset T of size $|T| > \frac{|Z|}{3}$.

Proof.

Fix any set $Z = \{z_1 < \ldots < z_n\}$ of integers.

Let q be a prime number that is of the form $q = 3k + 2$ and that does not divide any number in Z (e.g., choose q strictly larger than the absolute values of all numbers in Z).

Sum-free subsets

Proof (continued).

Pick r in $N_q := \{1, \ldots, q - 1\}$ uniformly at random .

For $i = 1, \ldots, n$, pick $d_i \in N_q$ such that $d_i \equiv z_i \cdot r \bmod q$.

Each d_i is uniformly distributed in N_q because by primality of q, for any fixed z the mapping $r \mapsto z \cdot r$ is a bijection of N_q.

Let $M = \{k + 1, \ldots, 2k + 1\}$ und $T_r = \{z_i : d_i \in M\}$.

The set T_r ist sum-free, because

$$z_{i_1} + z_{i_2} = z_{i_3} \quad \text{implies} \quad d_{i_1} + d_{i_2} \equiv d_{i_3} \bmod q \, ,$$

where the latter cannot hold for $d_{i_1}, d_{i_2}, d_{i_3} \in M$. Furthermore,

$$\mathbf{E}[|T_r|] = \sum_{z_i \in Z} \mathbf{P}[d_i \in M] = |Z| \frac{|M|}{|N_q|} = |Z| \frac{k+1}{3k+1} > \frac{|Z|}{3} \, ,$$

hence for some choice of r we have $|T_r| > \frac{|Z|}{3}$. $\square$

The isolating lemma

Definition

A *weight function* on a set X is a function $w \colon X \to \mathbb{R}$. A weight function w on X extends to subsets of X via $w(S) = \sum_{a \in S} w(a)$.

Theorem (Isolating Lemma)

Let $X = \{a_1, \ldots, a_m\}$ be a set of size $m > 0$ and let $\boldsymbol{F} = \{S_1, \ldots, S_k\}$ be a nonempty set of subsets of X.
Let a weight function w on X be determined by choosing uniformly and independently for each x in X a value $w(x)$ in $\{1, \ldots, 2m\}$.
Then with probability at least $1/2$, the minimal weight

$$w_{\min} = \min_{S \in \boldsymbol{F}} w(S)$$

is attained by a unique set in $\boldsymbol{F}$.

The isolating lemma

Proof.

Call a set $S \in \boldsymbol{F}$ a *minimum weight set* if $w(S)$ is equal to $w_{\min}$.

Furthermore, call a member a of X *ambiguous* if there are two minimum weight sets S^- and S^+ such that $a \notin S^-$ and $a \in S^+$.

Observe that there is a unique minimum weight set if and only if no member of X is ambiguous.

It suffices to show that any given member of X is ambiguous with probability at most $1/2m$, because then the probability that X has an ambiguous member at all is at most $1/2$.

Fix any a in X and let

$$\boldsymbol{F}^- = \{S \in \boldsymbol{F} \colon a \notin S\}, \qquad \boldsymbol{F}^+ = \{S \in \boldsymbol{F} \colon a \in S\} \, .$$

If one of the sets $\boldsymbol{F}^-$ or $\boldsymbol{F}^+$ is equal to $\boldsymbol{F}$, then a is never ambiguous, hence we can assume that both sets are nonempty.

The isolating lemma

Proof (continued).

Assuming that $w(x)$ has been determined for all x in $X \setminus \{a\}$, let

$$w^- = \min_{S \in \boldsymbol{F}^-} \sum_{x \in S} w(x) \qquad \text{and} \qquad w^+ = \min_{S \in \boldsymbol{F}^+} \sum_{x \in S \setminus \{a\}} w(x)\,,$$

and let $d = w^- - w^+$ (where d may be negative).

Then $w_{\min} = w^-$ or $w_{\min} = w^+ + w(a)$, and both equations hold simultaneously if and only if $w(a) = d$.

In case $w(a) < d$, all minimum weight sets are in $\boldsymbol{F}^+$.
In case $w(a) > d$, all minimum weight sets are in $\boldsymbol{F}^-$.
In both cases, a cannot be ambiguous by definition of $\boldsymbol{F}^-$ and $\boldsymbol{F}^+$.

Hence a can only be ambiguous in case $w(a) = d$, where the latter has probability at most $1/2m$ because $w(a)$ is chosen uniformly and independently of the other weights from a set of size $2m$. ☐

Crossing numbers

Planar graphs

- In what follows, the term graph refers to an undirected graph that is simple, i.e., does neither have loops nor multiple edges.
- A graph is planar if the graph can be embedded into the plane without crossing edges.

  (We will use the notion of an embedding and other notation such as face of an embedding without defining them and refer to Diestel's monograph *Graph Theory*.)
- Furthermore, we will assume certain properties of planar graphs and their embeddings that are intuitively clear without giving formal proofs.

  For example, given an embedding of a planar graph G into the plane, for any cycle (i.e., closed path) in G, part of the plane is inside and part is outside the cycle and no face of the embedding can intersect both parts of the cycle.

Crossing numbers

Theorem (Euler's formula)

Let G be a connected planar graph with n nodes and m edges such that G has an embedding in the plance with f faces (including the outer face). Then holds $f - m + n = 2$.

Sketch of proof.

Use induction on the number of cycles in G.

A graph without cycles is a tree, hence has $m + 1$ edges and a single face and Euler's formula holds.

For a plane graph with $k > 0$ cycles, removing an edge from a cycle decreases both the number of edges and the number of faces by 1, while for the resulting graph Euler's formula holds by the induction hypothesis.

Corollary

All embeddings of a planar graph have the same number of faces.

Crossing numbers

Proposition

Let G be a planar graph with n nodes and $m > 1$ edges.
Then holds $m \leq 3n - 6$.

Proof.

It suffices to prove the assertion for connected G because any nonconnected planar graph can be transformed into a connected planar graph by adding edges.

We can assume $m \geq 3$ because G is simple, hence for $m = 2$ we have $n = 3$ and the assertion is true.

Let f_i be the number of faces of G that are bounded by i edges where edges that are "surrounded" by a face are counted twice.

That is, we view an edge as having two "sides" and when counting the number of edges that bound a face in fact we are counting the sides of edges that bound the face.

Crossing numbers

Proof (continued).

The graph G is simple and $m \geq 3$, hence $f_1 = f_2 = 0$.

By counting the edges in two different ways, we obtain

$$\begin{aligned} f &= \ \ f_3 + \ \ f_4 + \ \ f_5 + \dots , \\ 2m &= 3f_3 + 4f_4 + 5f_5 + \dots , \end{aligned}$$

hence we have $0 \leq 2m - 3f$ and Euler's formula yields

$$0 \leq 2m - 3f = 2m - 3(m - n + 2) = 3n - 6 - m. \qquad \square$$

Crossing numbers

An embedding of a graph (V, E) into the plane consists of

- $|V|$ pairwise distinct points of the plane, where we idenify these points with the nodes in V,
- a curve with endpoints u and v for each edge $\{u, v\}$ in E, where we identify these curves with the edges in E.

We want to define a notion of crossing edges and, for a given graph, of embedding with a minimum number of crossings.

How should we count crossings of an embedding?

When are two crossings of an embedding identical?

An embedding with a minimum number of crossings shouldn't

- use "tricks" such as an edge intersecting a node that is not an endpoint of the edge,
- have unnecessary crossings such as in the case where two edges cross several times (see below).

Thus we want to define the notion of crossing and to count crossings in a way such that the number of crossings is increased by using such tricks and by unnecessary crossings.

Crossing numbers

Definition

Two edges of an embedding *cross* in a point if this point belongs to both edges but is not a common endpoint.

With an ordering of nodes understood, a *crossing* of an embedding is a set of two contiguous subcurves of two distinct edges such that

- the two edges cross in a point that is a common endpoint of the two subcurves,
- the other endpoints of the subcurves are the two lesser nodes of these edges, respectively.

In the simple case where for every pair of edges there is at most one crossing, we identify crossings with pair of edges.

Crossing numbers

Definition

The *crossing number* $\mathrm{cr}(G)$ of a graph G is the least k such that G has an embedding into the plane with at most k crossings.

An embedding of a graph G into the plane is *minimum* if the embedding has at most $\mathrm{cr}(G)$ crossings.

For a minimum embedding, the following assertions are true.

(i) No edge can cross itself.
(ii) Two edges that are incident to a common node cannot cross.
(iii) Two edges cannot cross twice or more.
(iv) Two edges cannot cross in an endpoint of one of the edges.

In any of the situations described in (i) through (iv), the given embedding can be transformed into an embedding of the same graph with strictly less crossings, hence for a minimum embedding, these situations cannot occur.

Crossing numbers

Proposition

For any graph G holds $\mathrm{cr}(G) \geq m - 3n + 6$.

Proof.

Transform a minimum embedding of a graph G into an embedding of a graph G' where each point where two edges of G cross is replaced by a new node, and the edges of the new graph correspond to minimum nonzero subcurves of the old edges that have two nodes of G' as endpoints.

By (i) through (iv), the graph G' is simple and has $n + \mathrm{cr}(G)$ nodes and $m + 2\mathrm{cr}(G)$ edges since every new node is distinct from the old nodes and has degree 4. The proposition shown above then yields

$$3(n + \mathrm{cr}(G)) - 6 \geq m + 2\mathrm{cr}(G), \qquad \text{hence}$$

$$\mathrm{cr}(G) \geq m - 3n + 6.$$

Crossing numbers

Theorem

Let G be a graph with n nodes and m edges and assume $m \geq 4n$. Then the crossing number of G is bounded from below as follows

$$\frac{1}{64}\frac{m^3}{n^2} \leq \mathrm{cr}(G) \,.$$

Proof (Aigner and Ziegler, *Proofs from the book*, Chapter 32).

Let p be a real number between 0 and 1 to be determined later.

Determine a set V_p of nodes of G by tossing a coin for each node of G such that any single node is put into V_p with probability p.

Let G_p be the subgraph of G that is induced by the set V_P.

Crossing numbers

Proof.

Let n_p and m_p be the number of nodes and edges of the graph G_p.

Fix a minimum embedding of G and consider the embedding of G_p that is obtained by restricting the given embedding of G to G_p.

Let x_p be the number of crossings in this embedding of G_p.

Observe that $x_p \geq \mathrm{cr}(G_p)$, hence by the proposition above we have

$$\mathbf{E}\left[x_p - m_p + 3n_p\right] \geq 0$$

As usual, we have $\mathbf{E}\left[n_p\right] = pn$ and $\mathbf{E}\left[m_p\right] = p^2 m$. Furthermore,

$$\mathbf{E}\left[x_p\right] = p^4 \mathrm{cr}(G) \,.$$

because a crossing in the given embedding of G is also in the new embedding if and only if all 4 pairwise distinct endpoints of two corresponding crossing edges are in G_p.

Crossing numbers

Proof (continued).

By linearity of expectation, the preceding discussion yields

$$p^4 \mathrm{cr}(G) - p^2 m + 3pn = \mathbf{E}\left[x_p\right] - \mathbf{E}\left[m_p\right] + 3\mathbf{E}\left[x_p\right] \geq 0, \text{ hence}$$

$$\mathrm{cr}(G) \geq \frac{p^2 m - 3pn}{p^4} \,.$$

Now let $p = \frac{4n}{m}$. By assumption $p \leq 1$, and we obtain

$$\mathrm{cr}(G) \geq \frac{pm - 3n}{p^3} = \frac{4n - 3n}{p^3} = \frac{1}{64}\frac{m^3}{n^2} \,. \qquad \square$$

Points on a circle

Proposition

If n points on a circle are chosen uniformly and mutually independently, then with probability $1 - \frac{2n}{2^n}$ the convex hull of these points contains the center of the circle.

Proof.

Fix any circle. For a point x on the circle let the mirror point of x be the unique point that differs from x and is on the line through x and the center of the circle.

If a set P of n points is determined by a chance experiment where

(i) n points on the circle are chosen uniformly and mutually independent,

(ii) for each such point a fair coin is tossed in order to decide whether the point or its mirror point is put into P,

then the points in P are actually chosen uniformly and mutually independently.

Points on a circle

Proof (continued).

Let X be the set of all points chosen in Step i together with their mirror points (we can assume that all these points are pairwise distinct because with probability 1 this is indeed the case).

Call a subset of X a candidate set if the set contains for each point in X either the point or its mirror point.

There are exactly 2^n candidate sets.

The set P is uniformly distributed in the set of candidate sets.

Call a candidate set one-sided if there is a line through the center of the circle such that the candidate set is contained in one of the half-planes determined by the line.

The center of the circle is not in the convex hull of the points in P if and only if P is one-sided.

There are exactly $2n$ candidate sets that are one-sided, hence the probability that P is one-sided is $\frac{2n}{2^n}$. ☐