

Magnetic domains in thin ferromagnetic films with strong perpendicular anisotropy

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February 6, 2017

Abstract

We investigate the scaling of the ground state energy and optimal domain patterns in thin ferromagnetic films with strong uniaxial anisotropy and the easy axis perpendicular to the film plane. Starting from the full three-dimensional micromagnetic model, we identify the critical scaling where the transition from single domain to multidomain ground states such as bubble or maze patterns occurs. Furthermore, we analyze the asymptotic behavior of the energy in two regimes separated by a transition. In the single domain regime, the energy Γ -converges towards a much simpler two-dimensional and local model. In the second regime, we derive the scaling of the minimal energy and deduce a scaling law for the typical domain size.

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1 Introduction

Ferromagnetic materials are an important class of solids which have played an indispensable role in data storage technologies of the digital age [51, 22, 64]. Their utility for technological applications stems from the basic physical property of ferromagnets to exhibit spatially ordered magnetization patterns – magnetic domains – under a variety of conditions [31]. The mechanisms behind the magnetic domain formation can be quite complex, but usually domain patterns may be understood from the energetic considerations based on the micromagnetic modeling framework [31, 11, 20]. Starting with the early works of Landau and Lifshitz [46] and Kittel [38], ground states of various ferromagnetic systems have been the subject of extensive studies in the physics community (see [31] and references therein), and more recently in the mathematical literature (for a review, see [20]). In particular, within the micromagnetic framework the ground state domain structure of macroscopically thick uniaxial ferromagnetic films is by now fairly well understood mathematically in terms of the energy and length scales, as well as some of the qualitative properties of the domains [14, 15, 17, 39, 58]. In contrast, apart from only a handful of studies [27, 16, 50, 54], the vast majority of mathematical treatments of microscopically thin ferromagnetic films deal with the situation in which the magnetization prefers to lie in the film plane (see, e.g., [25, 12, 19, 52, 41, 18, 44, 34, 33, 32, 13]; this list is certainly not complete). Thus, one of the fundamental open problems in the

theory of uniaxial ferromagnets is to rigorously characterize their ground states in the case of films of vanishing thickness when the magnetization prefers to align normally to the film plane (for various ansatz-based computations in the physics literature, see [42, 21, 37, 57]). This problem is the main subject of the present paper.

Recent advances in nanofabrication allow an unprecedented degree of spatial resolution, with features of only a few atomic layers in thickness and tens of nanometers laterally for planar structures [65], enabling synthesis of ultrathin ferromagnetic films and multilayer structures with novel material properties. Over the last decade, there has been a major focus on films with thickness of only a few atomic layers, primarily due to their promising applications in spintronics [4]. One of the important features of these films is the emergence of perpendicular magnetocrystalline anisotropy due to the increased importance of surface effects [29, 35], favoring the magnetization vector to lie along the normal to the film plane. As a result, the magnetization may exhibit either stripe or bubble domain phases depending on the applied external field and other factors [63, 60, 30, 67, 56]. We note that studies of magnetic bubble domains in relatively thick films have a long history in the context of magnetic memory devices (see, e.g., [42] and the book [48]). However, the occurrence of additional physical effects in ultrathin films, such as spin transfer torque [9, 24, 36], Dzyaloshinskii-Moriya interaction [7, 59] and electric field-controlled perpendicular magnetic anisotropy [23, 49] allow for much greater manipulation of the domain patterns, resulting in a renewed attention to bubble domains from experimentalists [36, 43, 66, 61, 62]. In particular, the topological characteristics of the bubble domain patterns in these materials are of great current interest [10, 24, 55]. These considerations further motivate the present study of the basic problem noted at the end of the preceding paragraph.

In this paper, we are interested in deriving a reduced two-dimensional model for ultrathin ferromagnetic films with perpendicular anisotropy and using it to asymptotically characterize the observed ground states and, more generally, all low energy states in films of large spatial extent. Our starting point is the three-dimensional micromagnetic energy functional, coming from the continuum theory of uniaxial bulk ferromagnets [45]. In a partially non-dimensionalized form, the micromagnetic energy is given by

$$\mathcal{E}[m] = \int_{\Omega} (l_{\text{ex}}^2 |\nabla m|^2 + Q(m_1^2 + m_2^2) - 2h_{\text{ext}} \cdot m) \, d^3x + \int_{\mathbb{R}^3} |h|^2 \, d^3x. \quad (1.1)$$

In (1.1), $\Omega \subset \mathbb{R}^3$ denotes the region in space occupied by the ferromagnet and \mathcal{E} is minimized among all $m \in H^1(\Omega; \mathbb{S}^2)$. The stray field $h : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is determined by the static Maxwell's equations in matter

$$\nabla \cdot (h + m) = 0 \quad \text{and} \quad \nabla \times h = 0, \quad (1.2)$$

so that the energy density depends in a nonlocal way on m . Furthermore, $h_{\text{ext}} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ describes an external magnetic field. The exchange length l_{ex} and the non-dimensional quality factor Q are material parameters. For an introduction to micromagnetic modeling we refer to e.g. [31, 20]. Note that additional physical effects due to the film surfaces

may be easily incorporated and would lead to the same type of a reduced two-dimensional model [54].

Since our focus is on materials with perpendicular anisotropy, we assume that the parameter Q in (1.1) is greater than one (for a detailed explanation, see the following section). The high anisotropy leads to magnetizations that are predominantly perpendicular to the film plane. It is well-known that such materials feature magnetizations that consist of one or many regions of nearly constant magnetization, called *magnetic domains*, separated by interfaces, called *domain walls*. In this work, we identify the critical scaling for the size of the sample where a transition from single domain states to multidomain states occurs. Moreover, we analyze the asymptotic behavior of the energy in the two regimes separated by this transition. In the subcritical regime, the global minimizers are the single domain states $m = \pm e_3$. We derive the asymptotic behavior of the energy in this regime in the framework of Γ -convergence. The reduced energy turns out to be much simpler than the full energy, in particular, it is two-dimensional and local. In the supercritical regime, which lies beyond the transition towards multidomain configurations, we establish the scaling of the energy (up to a multiplicative constant) and characterize sequences that achieve this scaling. Our analysis shows that the magnetization in this regime consists of several domains and suggests that the typical distance between domain walls scales as

$$\text{typical domain size } S \sim \frac{e^{\frac{2\pi l_{\text{ex}}\sqrt{Q-1}}{T}}}{\sqrt{Q-1}} l_{\text{ex}} \quad (1.3)$$

where T is the thickness of the film.

We will show that in the regimes we consider the leading order of the micromagnetic energy, upon rescaling and subtracting a constant, is given by the following two-dimensional functional defined for $m \in H^1(\mathbb{T}^2; \mathbb{S}^2)$:

$$F_{\varepsilon, \lambda}[m] = \int_{\mathbb{T}^2} \left(\frac{\varepsilon}{2} |\nabla m|^2 + \frac{1}{2\varepsilon} (1 - m_3^2) \right) d^2x - \frac{\lambda}{|\log \varepsilon|} \int_{\mathbb{T}^2} |\nabla^{1/2} m_3|^2 d^2x. \quad (1.4)$$

In (1.4), $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ denotes the square flat torus of unit side length, and we have assumed periodicity to avoid boundary effects for simplicity (see also the next section), ε is the renormalized Bloch wall width and λ is the renormalized film thickness (see the following section for the precise definitions). We note that a similar result for a closely related problem of a Ginzburg-Landau energy with dipolar interactions has been obtained in [53], where the meaning of the asymptotic equivalence between the full energy of three-dimensional configurations and the reduced energy of their e_3 -averages is discussed in more detail.

The main part of our analysis is concerned with the asymptotic behavior of (1.4) as $\varepsilon \rightarrow 0$ for different values of $\lambda > 0$. Note that the last term in (1.4) occurs with a negative sign and hence prefers oscillations of m_3 . As it turns out, the value of the parameter λ is crucial - in fact, we will show that the asymptotic behavior changes at

$\lambda = \lambda_c$, where $\lambda_c = \frac{\pi}{2}$, which is a singular point in the terminology of [8]. For $\lambda < \lambda_c$ the Γ -limit $F_{*,\lambda} := \Gamma(L^1)\text{-}\lim_{\varepsilon \rightarrow 0} F_{\varepsilon,\lambda}$ measures the length of the interface separating regions with $m \approx e_3$ and $m \approx -e_3$ (see Theorem 3.5)

$$F_{*,\lambda}[m] = \begin{cases} \left(1 - \frac{\lambda}{\lambda_c}\right) \int_{\mathbb{T}^2} |\nabla m_3| \, d^2x, & \text{for } m \in BV(\mathbb{T}^2; \{\pm e_3\}), \\ +\infty, & \text{otherwise.} \end{cases} \quad (1.5)$$

Note that the last term in (1.4) leads to a reduction of the interfacial cost by $\frac{\lambda}{\lambda_c}$ compared to the classical result [3] for $\lambda = 0$. On the other hand, for $\lambda > \lambda_c$, the scaling of the minimal energy changes (see Theorem 3.6)

$$\min F_{\varepsilon,\lambda} \sim -\frac{\lambda \varepsilon^{\frac{\lambda_c - \lambda}{\lambda}}}{|\log \varepsilon|} \xrightarrow{\varepsilon \rightarrow 0} -\infty, \quad (1.6)$$

and sequences (m_ε) which achieve the optimal scaling $F_{\varepsilon,\lambda}[m_\varepsilon] \sim \min F_{\varepsilon,\lambda}$ are highly oscillatory in the sense that

$$\int_{\mathbb{T}^2} |\nabla (m_\varepsilon)_3| \, d^3x \sim \varepsilon^{\frac{\lambda_c - \lambda}{\lambda}} \xrightarrow{\varepsilon \rightarrow 0} +\infty. \quad (1.7)$$

Furthermore, for $\lambda \geq \lambda_c$, the leading order contributions of all three terms in (1.4) cancel. The main difficulty in the proof is to find asymptotically optimal estimates for the non-local term.

A reduction of the full three-dimensional micromagnetic energy to a local two-dimensional model in the thin film limit was first established rigorously in [27]. Subsequently, several thin film regimes for magnetically soft materials have been identified and analyzed, see e.g. [12, 19, 52, 41, 44, 33]. However, since we consider materials with high perpendicular anisotropy, our setting is considerably different, as we now explain. For thin films of the form $\Omega = \mathbb{T}^2 \times (0, t)$, the leading order contribution of the stray field energy penalizes the out-of-plane component of the magnetization. Neglecting boundary effects, we have (see e.g. Theorem 6.2)

$$\left| \int_{\mathbb{T}^2 \times \mathbb{R}} |h|^2 \, d^3x - \int_{\mathbb{T}^2 \times (0,t)} m_3^2 \, d^3x \right| \lesssim t \int_{\mathbb{T}^2 \times (0,t)} |\nabla m|^2 \, d^3x.$$

To our knowledge, the first result in this direction is contained in [27]. In the absence of high perpendicular anisotropy or a sufficiently strong external field (as in the previously mentioned papers) the micromagnetic energy forces the out-of-plane component m_3 to vanish asymptotically. In our setting, the anisotropy energy $Q \int_{\Omega} (m_1^2 + m_2^2) \, d^3x = Q \int_{\Omega} (1 - m_3^2) \, d^3x$ is however sufficiently strong (recall that $Q > 1$) such that low energy configurations require $m \approx \pm e_3$ on most of the domain.

The behavior of the material changes when the film can no longer be considered to be thin. In [14] the scaling of the ground state energy was identified for the two-dimensional

micromagnetic model and in [15] for the three-dimensional model. Magnetizations with optimal energy involve so-called branching domain patterns which become finer and finer as they approach the boundary of the sample. When the ferromagnetic sample is exposed to a critical external field, a transition between a uniform and a branching domain pattern occurs. The critical field strength and the scaling of the micromagnetic energy for this regime were derived in [39]. In our regime, the thickness of the film is so small that this does not only exclude the branching patterns that occur in bulk samples, but actually forces the magnetization to become constant in the direction normal to the film plane.

Notation: For $x \in \mathbb{R}^3$ we write $x = (x', x_3)$, where x' is the projection of x onto the first two components. The square flat torus with side length $\ell > 0$ is denoted by $\mathbb{T}_\ell^2 := (\mathbb{R}^2/\ell\mathbb{Z}^2)$, and we abbreviate $\mathbb{T}^2 := \mathbb{T}_1^2$. We frequently identify functions $u : \mathbb{T}_\ell^2 \rightarrow \mathbb{R}$ with periodic functions $v : \mathbb{R}^2 \rightarrow \mathbb{R}$ by means of the natural projection $p : \mathbb{R}^2 \rightarrow \mathbb{T}_\ell^2$, i.e. $u = v \circ p$.

For $u \in L^1(\mathbb{T}_\ell^2 \times (0, t))$ we write $\bar{u} \in L^1(\mathbb{T}_\ell^2)$ to denote the e_3 -average, given by

$$\bar{u}(x') = \frac{1}{t} \int_0^t u(x', x_3) dx_3. \quad (1.8)$$

Moreover, for every $v \in L^1(\mathbb{T}_\ell^2)$ we write $\chi_{(0,t)}v \in L^1(\mathbb{T}_\ell^2 \times (0, t))$ to denote the function $(\chi_{(0,t)}v)(x', x_3) = \chi_{(0,t)}(x_3)v(x')$.

Unless stated otherwise, the expression $f(x) \lesssim g(x)$ means that there exists a universal constant $C > 0$ such that the inequality $f(x) \leq Cg(x)$ holds for every x . The symbol \gtrsim is defined analogously with \geq instead of \leq and we write \sim if both \lesssim and \gtrsim hold.

For future reference, we now fix the constants in the definition of the Fourier coefficients. For $f \in L^2(\mathbb{T}_\ell^2)$, we write

$$\widehat{f}_k = \int_{\mathbb{T}_\ell^2} e^{-ik \cdot x} f(x) dx, \quad \text{where } k \in \frac{2\pi}{\ell} \mathbb{Z}^2. \quad (1.9)$$

The inverse Fourier transform is then given by

$$f(x) = \frac{1}{\ell^2} \sum_{k \in \frac{2\pi}{\ell} \mathbb{Z}^2} e^{ik \cdot x} \widehat{f}_k, \quad (1.10)$$

where convergence is understood in the $L^2(\mathbb{T}_\ell^2)$ sense. Parseval's Theorem then states that

$$\int_{\mathbb{T}_\ell^2} f^*(x)g(x) dx = \frac{1}{\ell^2} \sum_{k \in \frac{2\pi}{\ell} \mathbb{Z}^2} \widehat{f}_k^* \widehat{g}_k \quad \text{for } f, g \in L^2(\mathbb{T}_\ell^2), \quad (1.11)$$

where “*” denotes complex conjugation. Furthermore, we use the symbol $\nabla^s u$ to denote

$$\int_{\mathbb{T}_\ell^2} |\nabla^s u|^2 \, d^2x := \frac{1}{\ell^2} \sum_{k \in \frac{2\pi}{\ell} \mathbb{Z}^2} |k|^{2s} |\widehat{u}_k|^2 \quad (1.12)$$

for $s \in \mathbb{R}$. For $s = 1/2$ we will also use the following well-known real space representation of the (square of the) homogeneous $H^{1/2}(\mathbb{T}_\ell^2)$ -norm

$$\int_{\mathbb{T}_\ell^2} |\nabla^{1/2} u|^2 \, d^2x = \frac{1}{4\pi} \int_{\mathbb{T}_\ell^2} \int_{\mathbb{R}^2} \frac{|u(x+y) - u(x)|^2}{|y|^3} \, d^2y \, d^2x. \quad (1.13)$$

For the convenience of the reader, a proof is contained in the appendix.

Lastly, with the usual abuse of notation, for $\varepsilon \rightarrow 0$ we will refer to $(m_\varepsilon) \in H^1(\mathbb{T}_\ell^2; \mathbb{S}^2)$ as a sequence, implying the sequence of $m_{\varepsilon_k} \in H^1(\mathbb{T}_\ell^2; \mathbb{S}^2)$ for some sequence of $\varepsilon_k \rightarrow 0$ as $k \rightarrow \infty$. Similarly, when dealing with the family of functionals $\{F_{\varepsilon, \lambda}\}$ we are always dealing with sequences $F_{\varepsilon_k, \lambda}$.

2 Setting

In order to non-dimensionalize the micromagnetic energy, we express lengths as multiples of the exchange length l_{ex} and rescale (effectively this amounts to setting $l_{\text{ex}} = 1$). We are interested in thin ferromagnetic films of uniform (non-dimensionalized) thickness t . For simplicity, we assume that the film extends infinitely in the film plane and that its magnetization is periodic in both in-plane coordinates with period ℓ . This means that we neglect boundary effects in the case of a finite sample of large spatial extent.

The film is composed of a uniaxial ferromagnetic material whose easy axis is perpendicular to the film plane, i.e. parallel to e_3 . Furthermore, we assume that the external field h_{ext} is parallel to e_3 and hence independent of x_3 (due to $\nabla \cdot h_{\text{ext}} = 0$). By a slight abuse of notation, from now on, we consider $h_{\text{ext}} : \mathbb{T}_\ell^2 \rightarrow \mathbb{R}$ as a scalar function. The non-dimensionalized energy per unit-cell $\mathbb{T}_\ell^2 \times (0, t)$ then reads

$$E[m] := \int_{\mathbb{T}_\ell^2 \times (0, t)} (|\nabla m|^2 + Q(m_1^2 + m_2^2) - 2m_3 h_{\text{ext}}) \, d^3x + \int_{\mathbb{T}_\ell^2 \times \mathbb{R}} |h|^2 \, d^3x. \quad (2.1)$$

In the last term of (2.1), the stray field is the unique distributional solution $h \in L^2(\mathbb{T}_\ell^2 \times \mathbb{R}; \mathbb{R}^3)$ of

$$\nabla \times h = 0 \quad \text{and} \quad \nabla \cdot (h + m) = 0 \quad \text{in } \mathbb{T}_\ell^2 \times \mathbb{R}, \quad (2.2)$$

where $m \in H^1(\mathbb{T}_\ell^2 \times (0, t))$ is extended by zero to $\mathbb{T}_\ell^2 \times \mathbb{R}$. Hence, up to a sign, h equals the Helmholtz projection of m onto the space of gradients. We also use the notation $h = h[m]$ to denote the solution of (2.2).

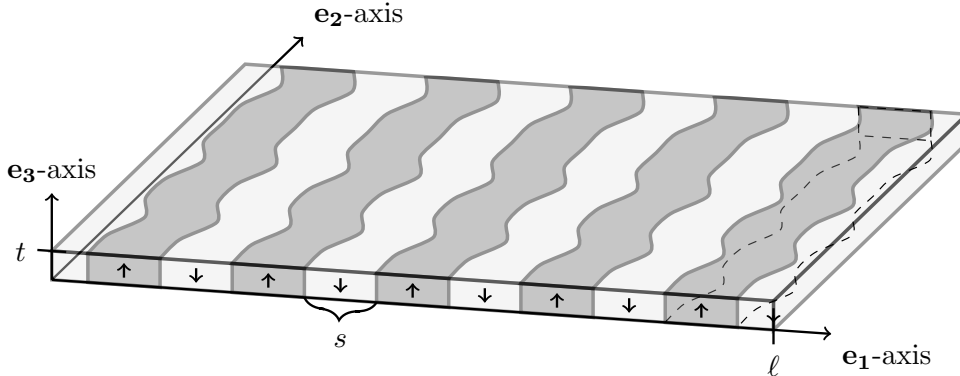


Figure 1: Typical magnetization pattern ("stripe pattern") in a unit cell $\mathbb{T}_\ell^2 \times (0, t)$ of the ferromagnetic film. The arrows represent the value of the magnetization $m(x)$ at x , which is approximately constant across regions of the same color. The domains are separated by continuous domain walls of vanishing thickness, depicted as lines.

Note that (2.1) depends on the three dimensionless parameters ℓ, t and Q . We are interested in the asymptotic behavior of the energy in (2.1) for thin films (i.e. $t \ll 1$) with large extension in the film plane (i.e. $\ell \gg 1$) and high anisotropy (i.e. $Q > 1$).

2.1 Identification of the regimes and the reduced energy F

In this section, we motivate the rigorous results contained in section 3. We use heuristic arguments to identify the scaling of the transition between monodomain and multi-domain states, and to explain how the micromagnetic energy E in (2.1) is related to the two-dimensional reduced energy F in (1.4). Roughly speaking, we will argue that (upon rescaling) F is a prototype for the next-to-leading-order term in the Γ -development of E , cf. [2].

To simplify the exposition, we neglect the energy contribution due to the external field h_{ext} . Furthermore we make two assumptions (for this section only), stated below. These assumptions are actually consequences of the thin film regime (see (6.2) and Theorem 6.2). Our assumptions are:

- (i) The magnetization m is constant in the direction normal to the film, i.e.

$$m(x', x_3) = \chi_{(0,t)}(x_3) \bar{m}(x') \quad \text{for } x = (x', x_3) \in \mathbb{T}_\ell^2 \times (0, t). \quad (\text{i})$$

- (ii) The stray field energy can be approximated by

$$\int_{\mathbb{T}_\ell^2 \times \mathbb{R}} |h[m]|^2 d^3x \approx t \int_{\mathbb{T}_\ell^2} \bar{m}_3^2 d^2x - \frac{t^2}{2} \int_{\mathbb{T}_\ell^2} |\nabla^{1/2} \bar{m}_3|^2 d^2x. \quad (\text{ii})$$

Assumption (i) can be understood as a consequence of the vanishing thickness of the film which is smaller than the thickness of optimal domain walls (so-called Bloch walls).

We will now motivate Assumption (ii). For magnetizations that are constant in the normal direction of the film, i.e. $m(x', x_3) = \chi_{(0,t)}(x_3)\bar{m}(x')$, it is well-known that the stray field energy splits into a contribution due to the normal component \bar{m}_3 and a contribution due to the in-plane divergence $\nabla' \cdot \bar{m}' = \partial_1 \bar{m}_1 + \partial_2 \bar{m}_2$, see e.g. [1, 25]. With the aid of the Fourier transform, a direct calculation yields (see also Theorem 6.2)

$$\begin{aligned} \int_{\mathbb{T}_\ell^2 \times \mathbb{R}} |h[m]|^2 d^3x &= \frac{1}{\ell^2} \sum_{k \in \frac{2\pi}{\ell} \mathbb{Z}^2} t \sigma(t|k|) |\widehat{\bar{m}}_{3,k}|^2 \\ &+ \frac{1}{\ell^2} \sum_{k \in \frac{2\pi}{\ell} \mathbb{Z}^2} t (1 - \sigma(t|k|)) \left| \frac{k}{|k|} \cdot \widehat{\bar{m}}'_k \right|^2, \end{aligned} \quad (2.3)$$

where the Fourier multiplier σ is given by $\sigma(s) = \frac{1-e^{-s}}{s}$. In the electrostatics analogy, the first term on the right hand side can be understood as the contribution of surface charges proportional to \bar{m}_3 at the top and bottom surface of the film, whereas the second term describes the contribution due to volume charges proportional to $\nabla' \cdot \bar{m}'$. Since the strong anisotropy requires $|m_3| \approx 1$ on most of the domain, a scaling argument indicates that only the contribution due to m_3 is relevant. Indeed, since $|1 - \sigma(t|k|)| \leq t|k| \leq t(1 + |k|^2)$ the contribution due to m' may be estimated by the exchange and anisotropy energy at lower order

$$\frac{1}{\ell^2} \sum_{k \in \frac{2\pi}{\ell} \mathbb{Z}^2} t (1 - \sigma(t|k|)) \left| \frac{k}{|k|} \cdot \widehat{\bar{m}}'_k \right|^2 \leq t^2 \int_{\mathbb{T}_\ell^2} (|\nabla m|^2 + |m'|^2) d^2x. \quad (2.4)$$

The right hand side of (ii) is obtained by neglecting the second term on the right hand side of (2.3) and approximating $\sigma(s) \approx 1 - \frac{s}{2}$ in the first term (see Theorem 6.2 for a rigorous version).

With (i), (ii) and $h_{\text{ext}} = 0$, the energy (2.1) can now be written as

$$E[m] \approx t \int_{\mathbb{T}_\ell^2} (|\nabla \bar{m}|^2 + Q (\bar{m}_1^2 + \bar{m}_2^2)) d^2x + t \int_{\mathbb{T}_\ell^2} \bar{m}_3^2 dx - \frac{t^2}{2} \int_{\mathbb{T}_\ell^2} |\nabla^{1/2} \bar{m}_3|^2 d^2x. \quad (2.5)$$

We use the constraint $|\bar{m}| = 1$ to combine the leading order stray-field energy term with the anisotropy energy

$$\int_{\mathbb{T}_\ell^2} \bar{m}_3^2 d^2x + \int_{\mathbb{T}_\ell^2} Q (\bar{m}_1^2 + \bar{m}_2^2) d^2x = \ell^2 + \int_{\mathbb{T}_\ell^2} (Q - 1) (\bar{m}_1^2 + \bar{m}_2^2) d^2x. \quad (2.6)$$

Inserting (2.6) into (2.5) allows to extract the leading order constant

$$E[m] \approx \ell^2 t + t \left(\int_{\mathbb{T}_\ell^2} (|\nabla \bar{m}|^2 + (Q - 1) (\bar{m}_1^2 + \bar{m}_2^2)) d^2x - \frac{t}{2} \int_{\mathbb{T}_\ell^2} |\nabla^{1/2} \bar{m}_3|^2 d^2x \right). \quad (2.7)$$

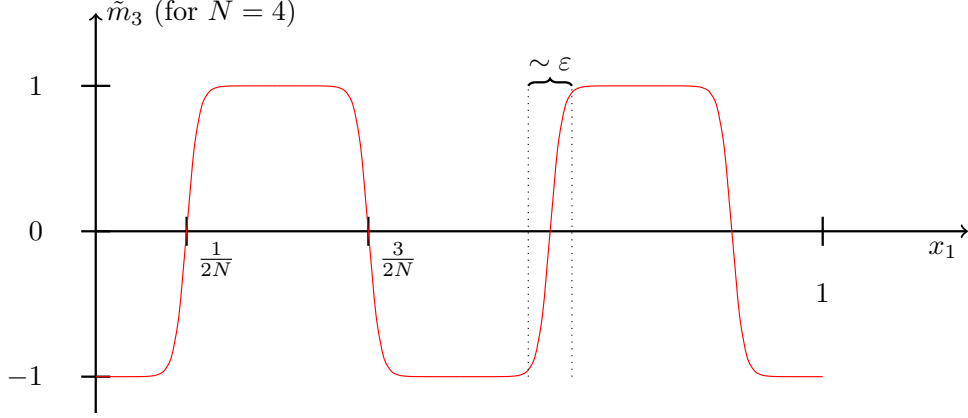


Figure 2: One-dimensional ansatz modeling a stripe pattern.

Upon rescaling \mathbb{T}_ℓ^2 to the fixed domain \mathbb{T}^2 and renormalizing the energy, we obtain

$$\begin{aligned} \frac{E[m(\ell)] - \ell^2 t}{\ell t \sqrt{Q-1}} &\approx \int_{\mathbb{T}^2} \left(\frac{1}{\ell \sqrt{Q-1}} |\nabla \bar{m}|^2 + \ell \sqrt{Q-1} (\bar{m}_1^2 + \bar{m}_2^2) \right) d^2 x \\ &\quad - \frac{t}{2\sqrt{Q-1}} \int_{\mathbb{T}^2} |\nabla^{1/2} \bar{m}_3|^2 d^2 x. \end{aligned} \quad (2.8)$$

In order to determine the critical scaling where minimizers of (2.8) cease to be constant and start to oscillate, we ask for which ℓ, t and Q it is possible to control the last term by the first integral

$$\frac{t}{2\sqrt{Q-1}} \int_{\mathbb{T}^2} |\nabla^{1/2} \bar{m}_3|^2 d^2 x \stackrel{?}{\lesssim} \int_{\mathbb{T}^2} \left(\frac{1}{\ell \sqrt{Q-1}} |\nabla \bar{m}|^2 + \ell \sqrt{Q-1} (\bar{m}_1^2 + \bar{m}_2^2) \right) d^2 x. \quad (2.9)$$

We make a one-dimensional ansatz \tilde{m} corresponding to N domains separated by smooth domain walls of width ε , see Figure 2. For the nonlocal term, a straightforward computation yields (see Lemma 5.2)

$$\int_{\mathbb{T}^2} |\nabla^{1/2} \tilde{m}_3|^2 d^2 x = \frac{1}{4\pi} \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \frac{|\tilde{m}_3(x+z) - \tilde{m}_3(x)|^2}{|z|^3} d^2 z d^2 x \approx \frac{4}{\pi} \log\left(\frac{1}{\varepsilon N}\right) N. \quad (2.10)$$

Since the nonlocal term depends only logarithmically on the transition layer, we optimize the width and internal structure of the transition layer for the first two terms in the energy by choosing $\varepsilon = \frac{1}{\ell \sqrt{Q-1}}$. For the corresponding Bloch wall profiles [31], we obtain

$$\int_{\mathbb{T}^2} \left(\frac{1}{\ell \sqrt{Q-1}} |\nabla \tilde{m}|^2 + \ell \sqrt{Q-1} (\tilde{m}_1^2 + \tilde{m}_2^2) \right) d^2 x \approx 2 \int_{\mathbb{T}^2} |\nabla \tilde{m}_3| d^2 x \approx 4N. \quad (2.11)$$

Hence

$$\frac{E[\tilde{m}(\ell \cdot)] - \ell^2 t}{\ell t \sqrt{Q-1}} \approx N \left(4 - \frac{2t}{\pi \sqrt{Q-1}} \log \left(\frac{\ell \sqrt{Q-1}}{N} \right) \right). \quad (2.12)$$

The (renormalized) energy of our ansatz (2.12) becomes negative, i.e. smaller than the energy of the constant configurations $m \equiv \pm e_3$, if $8\sqrt{Q-1} < \frac{4}{\pi} t \log \left(\frac{\ell \sqrt{Q-1}}{N} \right)$. By monotonicity in N , we expect that the critical scaling occurs for $N = 1$ and $t \sim t_c$, where

$$t_c \approx \frac{2\pi \sqrt{Q-1}}{\log(\ell \sqrt{Q-1})} \quad (2.13)$$

is the critical thickness of the onset of multidomain states.

Inserting (2.13) into (2.8) and abbreviating

$$\varepsilon = \frac{1}{\ell \sqrt{Q-1}}, \quad \lambda = \frac{t \log(\ell \sqrt{Q-1})}{4\sqrt{Q-1}}, \quad (2.14)$$

we are led to study the asymptotic behavior for $\varepsilon \rightarrow 0$ of the family of functionals $F_{\varepsilon, \lambda} : L^1(\mathbb{T}^2; \mathbb{S}^2) \rightarrow \mathbb{R} \cup \{+\infty\}$, given by

$$F_{\varepsilon, \lambda}[m] = \begin{cases} \int_{\mathbb{T}^2} \left(\frac{\varepsilon}{2} |\nabla m|^2 + \frac{1}{2\varepsilon} (1 - m_3^2) \right) d^2x - \frac{\lambda}{|\log \varepsilon|} \int_{\mathbb{T}^2} |\nabla^{1/2} m_3|^2 d^2x & \text{if } m \in H^1(\mathbb{T}^2; \mathbb{S}^2), \\ +\infty & \text{otherwise,} \end{cases} \quad (2.15)$$

where $\lambda \sim 1$ is a fixed parameter and with $\min E \approx \ell^2 t + 2\ell t \sqrt{Q-1} \min F_{\varepsilon, \lambda}$.

Remark 2.1. (Natural cut-off in the stray field energy) For thin films, a natural approximation for the stray field energy is given by

$$\int_{\mathbb{T}_\ell^2 \times \mathbb{R}} |h[m_3 e_3]| d^3x \approx \int_{\mathbb{T}_\ell^2 \times (0, t)} m_3^2 d^3x - \frac{t^2}{8\pi} \int_{\mathbb{T}_\ell^2} \int_{\mathbb{R}^2 \setminus B_t} \frac{|\bar{m}_3(x+z) - \bar{m}_3(x)|^2}{|z|^3} d^2z d^2x, \quad (2.16)$$

i.e. the region $|z| \leq t$ is excluded in the last integral. However, our approximations in (ii) and in Theorem 6.2 ignore this cut-off. We will now explain, that due to periodicity, this cut-off is not relevant in our setting. Roughly speaking, the reason is that the length scale of the cut-off is much smaller than the width of domain walls, which is the smallest length scale on which m varies. More precisely, we have (see Lemma 4.1)

$$\begin{aligned} t^2 \int_{\mathbb{T}_\ell^2} \int_{B_t} \frac{|\bar{m}_3(x+z) - \bar{m}_3(x)|^2}{|z|^3} d^2x d^2z &\lesssim t^3 \int_{\mathbb{T}_\ell^2} |\nabla \bar{m}_3|^2 d^2x \\ &\lesssim t^2 \int_{\mathbb{T}_\ell^2 \times (0, t)} |\nabla m|^2 d^3x, \end{aligned} \quad (2.17)$$

so that the effect due to the cut-off is controlled by the exchange energy at lower order. Here we have implicitly used that the film is periodic and hence does not have boundaries. On the other hand, if the ferromagnetic material is modeled by a finite domain $\Omega \times (0, t)$, exploiting the cut-off in the stray field energy becomes crucial: At the boundary $\partial\Omega$, the out-of-plane component m_3 may have a jump so that $\|m_3\|_{H^{1/2}(\mathbb{R}^2)}$ would be infinite. Since the exchange energy is oblivious to this jump at the boundary, (2.17) does not hold for Ω instead of \mathbb{T}_ℓ^2 .

3 Main results and overview of the proof

Our main result is the identification of two thin-film regimes separated by a transition and the derivation of the asymptotic behavior of the energy in the regimes. We will state the results for the full energy E in Section 3.1 and for the reduced energy F in Section 3.2.

3.1 Results for the full energy E

In terms of ℓ , t and Q , the regimes may be expressed by

$$Q > 1, \quad \ell \gg 1 \quad \text{and} \quad \frac{t |\log(\ell\sqrt{Q-1})|}{4\sqrt{Q-1}} = \lambda \quad (3.1)$$

and $\lambda_c := \pi/2$, where

$\lambda < \lambda_c$ corresponds to the subcritical regime featuring single domain states,

$\lambda = \lambda_c$ corresponds to the transition,

$\lambda_c < \lambda < \gamma \frac{|\log(\ell\sqrt{Q-1})|}{Q-1}$, for some universal $\gamma > 0$, corresponds to the multidomain state.

The upper bound $\lambda < \gamma \frac{|\log(\ell\sqrt{Q-1})|}{Q-1}$ is necessary because in general magnetizations may not be approximately two-dimensional beyond this threshold.

It is convenient to rescale the domain of the ferromagnetic film to a fixed domain by means of the anisotropic transformation

$$\mathbb{T}_\ell^2 \times (0, t) \rightarrow \mathbb{T}^2 \times (0, 1) \quad \text{with} \quad (x_1, x_2, x_3) \mapsto \left(\frac{x_1}{\ell}, \frac{x_2}{\ell}, \frac{x_3}{t} \right), \quad (3.2)$$

and study the renormalized energy $J : L^1(\mathbb{T}^2 \times (0, 1); \mathbb{S}^2) \rightarrow \mathbb{R} \cup \{+\infty\}$, defined by

$$J[m] := \begin{cases} \frac{E[m(\ell \cdot, \ell \cdot, t \cdot)] - \ell^2 t}{\ell t \sqrt{Q-1}} & \text{for } m \in H^1(\mathbb{T}^2 \times (0, 1); \mathbb{S}^2), \\ +\infty & \text{otherwise.} \end{cases} \quad (3.3)$$

The asymptotic behavior of (3.3) in the subcritical regimes is characterized in the following theorem.

Theorem 3.1 (Subcritical regime). *Let $\lambda_c := \frac{\pi}{2}$, $\lambda \in [0, \lambda_c)$, $Q > 1$ and $(\ell_k, t_k, h_{\text{ext},k})_{k \in \mathbb{N}}$ be a sequence with*

$$\ell_k \rightarrow \infty, \quad \frac{t_k |\log(\ell_k \sqrt{Q-1})|}{4\sqrt{Q-1}} = \lambda \quad \text{and} \quad \frac{\ell_k}{\sqrt{Q-1}} h_{\text{ext},k}(\ell_k \cdot) \rightarrow g \quad (3.4)$$

for some $g \in L^1(\mathbb{T}^2)$ and for all $k \in \mathbb{N}$. Then the sequence of renormalized energies $\{J_k\}_{k \in \mathbb{N}}$, defined by (3.3) with $(\ell, t, h_{\text{ext}})$ replaced by $(\ell_k, t_k, h_{\text{ext},k})$, satisfies

(i) **Compactness:** For every sequence $(m_k) \in L^1(\mathbb{T}^2 \times (0, 1); \mathbb{S}^2)$ with

$$\limsup_{k \rightarrow \infty} J_k[m_k] < +\infty, \quad (3.5)$$

there exists a sub-sequence (not relabeled) and $\bar{m} \in BV(\mathbb{T}^2; \{\pm e_3\})$ such that

$$\int_{\mathbb{T}^2 \times (0, 1)} |m_k(x) - \bar{m}(x')| \, d^3x \rightarrow 0 \quad \text{for } k \rightarrow \infty. \quad (3.6)$$

(ii) **Γ -Convergence:** The sequence of functionals $\{J_k\}_{k \in \mathbb{N}}$ Γ -converges towards $J_* : L^1(\mathbb{T}^2; \{\pm e_3\}) \rightarrow \mathbb{R} \cup \{+\infty\}$ given by

$$J_*[\bar{m}] = \begin{cases} 2 \left(1 - \frac{\lambda}{\lambda_c}\right) \int_{\mathbb{T}^2} |\nabla \bar{m}_3| \, d^2x - 2 \int_{\mathbb{T}^2} g \bar{m}_3 \, d^2x & \text{if } \bar{m} \in BV(\mathbb{T}^2; \{\pm e_3\}), \\ +\infty & \text{otherwise.} \end{cases} \quad (3.7)$$

This means

(a) **liminf - Inequality:** Every sequence $(m_k) \in L^1(\mathbb{T}^2 \times (0, 1); \mathbb{S}^2)$ that converges towards $\bar{m} \in L^1(\mathbb{T}^2; \{\pm e_3\})$ in the sense of (3.6) satisfies

$$\liminf_{k \rightarrow \infty} J_k[m_k] \geq J_*[\bar{m}]. \quad (3.8)$$

(b) **Recovery Sequence:** For every $\bar{m} \in L^1(\mathbb{T}^2, \{\pm e_3\})$ there exists a sequence of magnetizations $(m_k) \in L^1(\mathbb{T}^2 \times (0, 1); \mathbb{S}^2)$ which converges towards \bar{m} in the sense of (3.6) and satisfies

$$\limsup_{k \rightarrow \infty} J_k[m_k] \leq J_*[\bar{m}]. \quad (3.9)$$

Whereas the energy favors single domain states in the subcritical regime, our next theorem shows that the energy leads to pattern formation in the supercritical regime.

Theorem 3.2 (Supercritical regime). *Let $h_{\text{ext}} = 0$. There are universal constants $\delta, K > 0$ such that for $Q, \ell, t > 0$ in the regime*

$$Q > 1, \quad t \leq \delta \min \left\{ \sqrt{Q-1}, \frac{1}{\sqrt{Q-1}} \right\} \quad \text{and} \quad \ell \geq K \frac{e^{2\pi t^{-1}\sqrt{Q-1}}}{\sqrt{Q-1}} \quad (3.10)$$

the minimal renormalized energy J in (3.3) scales as

$$-Ct\ell e^{-2\pi t^{-1}\sqrt{Q-1}} \leq \min J[m] \leq -ct\ell e^{-2\pi t^{-1}\sqrt{Q-1}}, \quad (3.11)$$

for some universal constants $0 < c < C$.

Furthermore, profiles achieving the optimal scaling in the regime (3.10) can be characterized as follows.

Proposition 3.3. *Let δ, K be as in Theorem 3.2, $h_{\text{ext}} = 0$ and let ℓ, t, Q satisfy (3.10). For any $\gamma > 0$ and all $m \in H^1(\mathbb{T}^2 \times (0, 1); \mathbb{S}^2)$ which satisfy*

$$J[m] \leq -\gamma t\ell e^{-2\pi t^{-1}\sqrt{Q-1}}, \quad (3.12)$$

we have

$$(i) \quad \int_{\mathbb{T}^2 \times (0, 1)} |m - \bar{m}|^2 \, d^3x \leq c_\gamma t^3 e^{-2\pi t^{-1}\sqrt{Q-1}} \sqrt{Q-1}, \quad (3.13)$$

$$(ii) \quad \int_{\mathbb{T}^2 \times (0, 1)} (m_1^2 + m_2^2) \, d^3x \leq c_\gamma e^{-2\pi t^{-1}\sqrt{Q-1}}, \quad (3.14)$$

$$(iii) \quad c_\gamma \ell e^{-2\pi t^{-1}\sqrt{Q-1}} \sqrt{Q-1} \leq \int_{\mathbb{T}^2} |\nabla' \bar{m}_3| \, d^2x \leq C_\gamma \ell e^{-2\pi t^{-1}\sqrt{Q-1}} \sqrt{Q-1} \quad (3.15)$$

$$(iv) \quad \int_{\mathbb{T}^2 \times (0, 1)} \left(\frac{|\nabla m|^2}{\ell \sqrt{Q-1}} + \ell \sqrt{Q-1} (1 - m_3^2) \right) \, d^3x - 2 \int_{\mathbb{T}^2} |\nabla \bar{m}_3| \, d^2x \\ \leq c_\gamma \frac{t}{\sqrt{Q-1}} \int_{\mathbb{T}^2} |\nabla \bar{m}_3| \, d^2x, \quad (3.16)$$

where $0 < c_\gamma < C_\gamma$ are constants (changing from line to line) which may depend only on γ .

We take a moment to interpret the statements (i)–(iv) in Proposition 3.3 above. Item (i) shows that the magnetization is approximately two-dimensional, i.e. independent of the thickness variable. Moreover, since $|m| = 1$, Item (ii) means that the magnetization is mostly perpendicular to the film (i.e. $m \approx \pm e_3$). Furthermore, Item (iii) is an estimate for the total length of the domain walls in a unit cell. Back in the original, physical variables, this quantity for the unit cell $(0, L)^2 \times (0, T)$ is

$$W := L \int_{\mathbb{T}^2} |\nabla \bar{m}_3| \, d^2x \stackrel{(3.15)}{\sim} \frac{L^2}{l_{\text{ex}}} e^{-\frac{2\pi l_{\text{ex}} \sqrt{Q-1}}{T}} \sqrt{Q-1}. \quad (3.17)$$

We expect that the stray field energy induces a repulsive interaction of (nearest) neighboring domain walls and leads to an approximately equidistant spacing of the walls. In view of (iii), the typical distance of neighboring walls should be

$$S := \frac{\text{length of the film}}{\# \text{ of walls on cross section}} \sim \frac{\ell}{\int_{\mathbb{T}^2} |\nabla' \bar{m}_3| \, d^2x} l_{\text{ex}} \stackrel{(3.15)}{\sim} \frac{l_{\text{ex}} e^{\frac{2\pi l_{\text{ex}} \sqrt{Q-1}}{t}}}{\sqrt{Q-1}}. \quad (3.18)$$

The exponential dependence of the typical distance between neighboring walls on the inverse thickness in (3.18) was already observed in ansatz-based computations in [37] for a two-dimensional sharp interface model. Item (iv) in Proposition 3.3 indicates that domain walls approximate Bloch walls of thickness proportional to $\varepsilon L = \frac{l_{\text{ex}}}{\sqrt{Q-1}}$ for which the left hand side of (3.16) is exactly zero. Note that (3.16) also implies that m approximately satisfies the optimal profile ODE in an L^2 -sense

$$\begin{aligned} \int_{\mathbb{T}^2 \times (0,1)} \left(\frac{|\nabla m_3|}{\sqrt{\ell \sqrt{Q-1} (1-m_3^2)}} - \sqrt{\ell \sqrt{Q-1} (1-m_3^2)} \right)^2 d^3x \\ \lesssim \frac{t}{\sqrt{Q-1}} \int_{\mathbb{T}^2} |\nabla m_3| \, d^2x, \end{aligned} \quad (3.19)$$

with the convention $\frac{|\nabla m_3|}{\sqrt{1-m_3^2}} = 0$ if $|m_3| = 1$. Finally, we want to mention that the estimate of the in-plane magnetization in Item (i) is consistent with the in-plane magnetization of a Bloch wall of length W (see (3.17)) and thickness $\frac{l_{\text{ex}}}{\sqrt{Q-1}}$.

Our third theorem addresses the transition where the cross-over from constant to non-constant global minimizers occurs and which separates the two previously described regimes.

Theorem 3.4 (Critical scaling). *Let $h_{\text{ext}} = 0$ and let $\delta > 0$ be as in Theorem 3.2. Then the following holds*

(i) **Cross-over of global minimizers** *There are constants $c, C > 0$ such that for ℓ, t, Q which satisfy*

$$Q > 1, \quad t \leq \delta \min \left\{ \sqrt{Q-1}, \frac{1}{\sqrt{Q-1}} \right\} \quad \text{and} \quad \ell \leq c \frac{e^{2\pi t^{-1} \sqrt{Q-1}}}{\sqrt{Q-1}} \quad (3.20)$$

the renormalized energy J is non-negative and $m \equiv \pm e_3$ are the only global minimizers, whereas for ℓ, t, Q which satisfy

$$Q > 1, \quad t \leq \delta \min \left\{ \sqrt{Q-1}, \frac{1}{\sqrt{Q-1}} \right\} \quad \text{and} \quad \ell \geq C \frac{e^{2\pi t^{-1} \sqrt{Q-1}}}{\sqrt{Q-1}} \quad (3.21)$$

the minimal rescaled energy $\min J$ is strictly negative and minimizers cannot be constant.

(ii) **Γ -convergence** For $\frac{t \log(\ell\sqrt{Q-1})}{\sqrt{Q-1}} = 2\pi$, J Γ -converges for $\ell\sqrt{Q-1} \rightarrow \infty$ towards

$$J_*[m] = \begin{cases} 0 & \text{if } m \in L^1(\mathbb{T}^2; \{\pm e_3\}), \\ +\infty & \text{otherwise.} \end{cases} \quad (3.22)$$

(iii) **Compactness upon rescaling** For $C > 0$ and $\ell\sqrt{Q-1} \rightarrow \infty$, sequences with

$$J[m] \leq \frac{C}{\log(\ell\sqrt{Q-1})} \quad (3.23)$$

are compact in $L^1(\mathbb{T}^2 \times (0, 1))$ with a limit of the form $\chi_{(0,1)} \bar{m}$ where $\bar{m} \in BV^1(\mathbb{T}^2; \{\pm e_3\})$.

3.2 Results for the simplified energy F

In this section, we will formulate results analogous to the ones in the previous section, but for the reduced energy F . The relation between the full energy E and the reduced two-dimensional energy F was explained heuristically in section 2.1 and will be made rigorous in section 6. The reason to formulate our results also in terms of F is mainly expositional: We believe that the main ideas are easier to understand when they are not obscured by additional difficulties arising from the reduction to a two-dimensional model and the stray-field energy approximation.

The behavior of the reduced energy F in the subcritical regime is summarized in the following theorem.

Theorem 3.5 (Subcritical regime). *Let $\lambda < \lambda_c := \frac{\pi}{2}$ and $F_{\varepsilon,\lambda}$ as defined in (2.15). Then the following holds*

(i) **Compactness:** *Every sequence (m_ε) in $H^1(\mathbb{T}^2; \mathbb{S}^2)$ with*

$$\limsup_{\varepsilon \rightarrow 0} F_{\varepsilon,\lambda}[m_\varepsilon] < +\infty \quad (3.24)$$

converges in $L^1(\mathbb{T}^2)$ (up to extracting a subsequence) towards a limit in $BV(\mathbb{T}^2; \{\pm e_3\})$.

(ii) **Γ -convergence:** *As $\varepsilon \rightarrow 0$, the family of functionals $\{F_{\varepsilon,\lambda}\}$ Γ -converges with respect to the $L^1(\mathbb{T}^2)$ -topology towards $F_{*,\lambda}$, given by*

$$F_{*,\lambda}[m] = \begin{cases} \left(1 - \frac{\lambda}{\lambda_c}\right) \int_{\mathbb{T}^2} |\nabla m_3| \, d^2x & \text{for } m \in BV(\mathbb{T}^2; \{\pm e_3\}) \\ +\infty & \text{otherwise.} \end{cases} \quad (3.25)$$

The next theorem is concerned with the minimal energy and the structure of low energy states in the supercritical regime.

Theorem 3.6 (Supercritical regime). *Let $\lambda_c := \frac{\pi}{2}$ and $F_{\varepsilon,\lambda}$ as defined in (2.15). There are constants $\delta < 1 < K$ such that for*

$$0 < \varepsilon < K^{-\frac{\lambda}{\lambda-\lambda_c}} \quad \text{and} \quad \lambda_c < \lambda < \delta |\log \varepsilon|, \quad (3.26)$$

the minimal energy of the family of functionals $\{F_{\varepsilon,\lambda}\}$ satisfies

$$-C \frac{\lambda \varepsilon^{\frac{\lambda_c-\lambda}{\lambda}}}{|\log \varepsilon|} \leq \min F_{\varepsilon,\lambda} \leq -c \frac{\lambda \varepsilon^{\frac{\lambda_c-\lambda}{\lambda}}}{|\log \varepsilon|} \quad (3.27)$$

for some universal constants $0 < c < C$. Moreover, the profiles achieving the optimal scaling can be characterized as follows. For any $\gamma > 0$ and all $m \in H^1(\mathbb{T}^2; \mathbb{S}^2)$ which satisfy

$$F_{\varepsilon,\lambda}[m] \leq -\gamma \frac{\lambda \varepsilon^{\frac{\lambda_c-\lambda}{\lambda}}}{|\log \varepsilon|}, \quad (3.28)$$

the quantities

$$\int_{\mathbb{T}^2} |\nabla m_3| \, d^2x \leq \int_{\mathbb{T}^2} \left(\frac{\varepsilon}{2} |\nabla m|^2 + \frac{1-m_3^2}{2\varepsilon} \right) \, d^2x \leq \frac{\lambda}{|\log \varepsilon|} \int_{\mathbb{T}^2} |\nabla^{1/2} m_3|^2 \, d^2x \quad (3.29)$$

agree to the leading order and scale as $\varepsilon^{\frac{\lambda_c-\lambda}{\lambda}}$, i.e. if A and B are any of the three quantities in (3.29), we have

$$c_\gamma \varepsilon^{\frac{\lambda_c-\lambda}{\lambda}} \leq A \leq C_\gamma \varepsilon^{\frac{\lambda_c-\lambda}{\lambda}} \quad \text{and} \quad |A - B| \leq \tilde{C}_\gamma \frac{\lambda}{|\log \varepsilon|} A, \quad (3.30)$$

for some positive constants c_γ, C_γ and \tilde{C}_γ which depend only on γ .

Under the assumptions of Theorem 3.6, statements analogous to (3.13) – (3.16) in Proposition 3.3 hold as well, they are simple consequences of the stronger statement (3.30).

The next theorem addresses the structure of minimizers in a neighborhood of the transition.

Theorem 3.7 (Critical scaling). *Let $\lambda_c := \frac{\pi}{2}$ and $F_{\varepsilon,\lambda_c}$ as defined in (2.15). Then the following holds*

(i) **Cross-over of global minimizers:** *There are two constants $0 < \beta_1 < 1 < \beta_2$ such that for*

$$\lambda \leq \lambda_-(\varepsilon) := \lambda_c \left(1 - \frac{|\log \beta_1|}{|\log \varepsilon|} \right) \quad (3.31)$$

the minimal energy $\min F_{\varepsilon,\lambda}$ is zero and only attained by the constant configurations $m \equiv \pm e_3$, whereas for

$$\lambda \geq \lambda_+(\varepsilon) := \lambda_c \left(1 + \frac{|\log \beta_2|}{|\log \varepsilon|} \right) \quad (3.32)$$

the minimal energy is strictly negative and minimizers cannot be constant.

(ii) **Γ -convergence:** As $\varepsilon \rightarrow 0$, the family of functionals $\{F_{\varepsilon, \lambda_c}\}$ Γ -converges with respect to the $L^1(\mathbb{T}^2)$ -topology towards F_{*, λ_c} , given by

$$F_{*, \lambda_c}[m] = \begin{cases} 0, & \text{if } m \in L^1(\mathbb{T}^2; \{\pm e_3\}) \\ +\infty & \text{otherwise,} \end{cases} \quad (3.33)$$

(iii) **Lack of compactness:** There is a sequence (m_ε) in $H^1(\mathbb{T}^2; \mathbb{S}^2)$ with

$$\limsup_{\varepsilon \rightarrow 0} F_{\varepsilon, \lambda_c}[m_\varepsilon] \rightarrow 0 \quad (3.34)$$

which is not precompact in $L^1(\mathbb{T}^2)$.

(iv) **Compactness upon rescaling:** For every $C > 0$, every sequence (m_ε) with

$$F_{\varepsilon, \lambda_c}[m_\varepsilon] \leq C |\log \varepsilon|^{-1} \quad (3.35)$$

converges in $L^1(\mathbb{T}^2)$ (up to extracting a subsequence) to a limit in $BV(\mathbb{T}^2; \{\pm e_3\})$.

Theorem 3.7 suggests that $|\log \varepsilon| F_{\varepsilon, \lambda_c}$ is the appropriate rescaling for the critical case. Unfortunately, it seems not possible to obtain the Γ -limit of $|\log \varepsilon| F_{\varepsilon, \lambda_c}$ with our $H^{1/2}$ -estimate (4.1) of the following section, because the constant c_* there is not optimal.

We illustrate our results in a phase diagram (Figure 3). It is not difficult to see that for each $0 < \varepsilon < 1$ there is a sharp threshold value $\lambda = \lambda_c(\varepsilon) > 0$ at which a transition from monodomain ($m \equiv +e_3$ or $m \equiv -e_3$) to multidomain ($m \neq \text{const}$) states as global energy minimizers occurs, with $\lambda_c(\varepsilon)$ a Lipschitz-continuous function on $[\delta, 1 - \delta]$ for every $0 < \delta < \frac{1}{2}$ (for the reader's convenience, a proof of this fact is presented in Lemma A.1 in the appendix). While we do not know the precise value of $\lambda_c(\varepsilon)$ for $\varepsilon > 0$, we show in Theorem 3.7 that $\lambda_-(\varepsilon) \leq \lambda_c(\varepsilon) \leq \lambda_+(\varepsilon)$ and $\lim_{\varepsilon \rightarrow 0} \lambda_c(\varepsilon) = \frac{\pi}{2}$, i.e. the definition above agrees with $\lambda_c := \lambda_c(0) = \frac{\pi}{2}$. Furthermore, global minimizers $m_{(\varepsilon, \lambda)}$ of $F_{\varepsilon, \lambda}$ with (ε, λ) between the red (dashed) curves of the form $\lambda(\varepsilon) = \lambda_c + \gamma |\log \varepsilon|^{-1}$ satisfy a uniform bound of the form $c \leq \int_{\mathbb{T}^2} |\nabla m_{(\varepsilon, \lambda), 3}|^2 dx \leq C$, with constants $C > c > 0$ depending only on the values of $\gamma > 0$ for these curves.

4 A bound on the homogeneous $H^{1/2}$ -norm

Since all three terms in F contribute in highest order to the limit, it is important to estimate the negative term $\int_{\mathbb{T}^2} |\nabla^{1/2} m_3|^2 dx$ with precise leading order constant. In this section we will establish an upper bound for the homogeneous $H^{1/2}$ -norm which is the key ingredient for the lower bounds (recall that the $H^{1/2}$ -term occurs in the energy with a negative sign).

We will prove the following

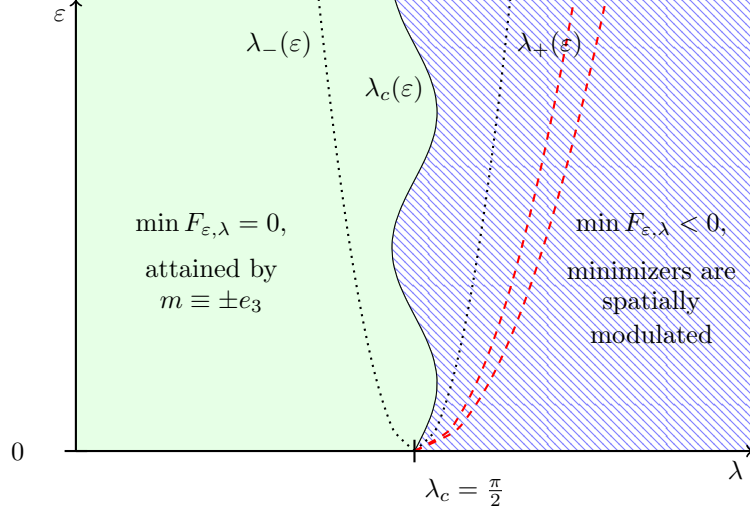


Figure 3: Sketch of the phase diagram for minimizers of $F_{\epsilon, \lambda}$ in terms of $\lambda > 0$ and $0 < \epsilon \ll 1$.

Lemma 4.1. *There is a universal constant $c_* \geq 1$ such that for every $f \in C^\infty(\mathbb{T}^2)$ and every $\epsilon > 0$ we have*

$$\int_{\mathbb{T}^2} |\nabla^{1/2} f|^2 \, d^2x \leq \frac{\epsilon}{2} \int_{\mathbb{T}^2} |\nabla f|^2 \, d^2x + \frac{2}{\pi} \log \left(c_* \max \left\{ 1, \min \left\{ \frac{\|f\|_\infty}{\epsilon \int_{\mathbb{T}^2} |\nabla f| \, dx}, \frac{1}{\epsilon} \right\} \right\} \right) \|f\|_\infty \int_{\mathbb{T}^2} |\nabla f| \, d^2x. \quad (4.1)$$

In Lemma 4.1, we improve an inequality established in [18]. Expressed in our setting, the inequality proved in [18] asserts that for every $\delta > 0$ there exists $M_\delta \gg 1$ such that for all $\epsilon \leq R$ and all $f : \mathbb{T}^2 \rightarrow \mathbb{R}$, we have

$$\sum_{k \in 2\pi\mathbb{Z}^2} \min \left\{ \frac{1}{\epsilon}, |k|, R|k|^2 \right\} |\widehat{f}_k|^2 \leq (1 + \delta) \frac{2}{\pi} \log \left(\frac{2M_\delta R}{\epsilon} \right) \|f\|_\infty \int_{\mathbb{T}^2} |\nabla f| \, d^2x. \quad (4.2)$$

Note that (4.1) implies a similar estimate

$$\int_{\mathbb{T}^2} |\nabla^{1/2} f|^2 \, d^2x \leq \frac{\epsilon}{2} \int_{\mathbb{T}^2} |\nabla f|^2 \, d^2x + \frac{2}{\pi} \log(c_*/\epsilon) \|f\|_\infty \int_{\mathbb{T}^2} |\nabla f| \, d^2x \quad (4.3)$$

for all $\epsilon \leq 1$, which is weaker than (4.1). Estimate (4.2) is an inequality for a regularized $\dot{H}^{1/2}$ -norm, whereas (4.3) estimates the full $\dot{H}^{1/2}$ -norm, but needs an additional \dot{H}^1 -term. It ceases to be optimal for functions which oscillate significantly. Indeed, let $\alpha \in (0, 1)$ and consider functions f with

$$\int_{\mathbb{T}^2} |\nabla f| \, d^2x \gtrsim \epsilon^{-\alpha} \|f\|_\infty. \quad (4.4)$$

Then the second term in (4.1) is smaller than the second term in (4.3) by a factor of $(1 - \alpha)$ for all f which satisfy (4.4). Asymptotic optimality in the case of strong oscillation is crucial to obtain the results on the supercritical regime.

The proof of Lemma 4.1 uses similar ideas as in [18] and is based on a separate treatment of distinct scales. However, our proof does not involve any Fourier Analysis.

Proof of Lemma 4.1. We will show that the following estimates hold for all $f \in C^\infty(\mathbb{T}^2)$ and all $0 < r \leq R$:

$$\int_{\mathbb{T}^2} \int_{B_r} \frac{|f(x+z) - f(x)|^2}{|z|^3} d^2z d^2x \leq \pi r \int_{\mathbb{T}^2} |\nabla f|^2 d^2x, \quad (4.5)$$

$$\int_{\mathbb{T}^2} \int_{B_R \setminus B_r} \frac{|f(x+z) - f(x)|^2}{|z|^3} d^2z d^2x \leq 8 \log(R/r) \|f\|_\infty \int_{\mathbb{T}^2} |\nabla f|^2 d^2x, \quad (4.6)$$

$$\int_{\mathbb{T}^2} \int_{\mathbb{R}^2 \setminus B_R} \frac{|f(x+z) - f(x)|^2}{|z|^3} d^2z d^2x \leq \frac{2\pi \|f\|_\infty}{R} \min \left\{ 4\|f\|_\infty, \int_{\mathbb{T}^2} |\nabla f|^2 d^2x \right\}. \quad (4.7)$$

The claim of the Lemma will follow by adding (4.5) – (4.7) and a suitable choice of r and R . Before we start with the proofs of estimates (4.5) – (4.7), we first record an auxiliary inequality for further use. By the Fundamental Theorem of Calculus, Jensen’s inequality and Fubini’s theorem we get

$$\begin{aligned} \int_{\mathbb{T}^2} |f(x+z) - f(x)|^p d^2x &= \int_{\mathbb{T}^2} \left| \int_0^1 \nabla f(x+sz) \cdot z ds \right|^p d^2x \\ &\leq \int_0^1 \int_{\mathbb{T}^2} |\nabla f(x+sz) \cdot z|^p d^2x ds \leq \int_{\mathbb{T}^2} |\nabla f(x) \cdot z|^p d^2x \end{aligned} \quad (4.8)$$

for all $z \in \mathbb{R}^2$ and all $1 \leq p < \infty$. In order to prove (4.5), we use Fubini’s Theorem and apply (4.8) with $p = 2$ to get

$$\int_{\mathbb{T}^2} \int_{B_r} \frac{|f(x+z) - f(x)|^2}{|z|^3} d^2z d^2x \stackrel{(4.8)}{\leq} \int_{B_r} \int_{\mathbb{T}^2} \frac{|\nabla f(x) \cdot z|^2}{|z|^3} d^2x d^2z. \quad (4.9)$$

We apply Fubini’s Theorem again and evaluate the integral with respect to z in polar coordinates

$$\begin{aligned} \int_{B_r} \int_{\mathbb{T}^2} \frac{|\nabla f(x) \cdot z|^2}{|z|^3} d^2x d^2z &= \left(\int_0^r \int_0^{2\pi} \cos^2 \phi d\phi d\rho \right) \left(\int_{\mathbb{T}^2} |\nabla f(x)|^2 d^2x \right) \\ &= \pi r \int_{\mathbb{T}^2} |\nabla f|^2 d^2x. \end{aligned} \quad (4.10)$$

Together, (4.9) and (4.10) yield the first estimate (4.5).

For the estimate involving intermediate distances (4.6), we use Fubini's Theorem (twice) and (4.8) with $p = 1$ to conclude

$$\int_{\mathbb{T}^2} \int_{B_R \setminus B_r} \frac{|f(x+z) - f(x)|^2}{|z|^3} d^2z d^2x \stackrel{(4.8)}{\leq} 2\|f\|_\infty \int_{\mathbb{T}^2} \int_{B_R \setminus B_r} \frac{|\nabla f(x) \cdot z|}{|z|^3} d^2z d^2x. \quad (4.11)$$

As in the proof of (4.2) in [18], we evaluate the inner integral in polar coordinates

$$\int_{B_R \setminus B_r} \frac{|\nabla f(x) \cdot z|}{|z|^3} d^2z = \int_r^R \int_0^{2\pi} \frac{|\nabla f(x)| |\cos \phi|}{\rho} d\phi d\rho = 4 \log \left(\frac{R}{r} \right) |\nabla f(x)|. \quad (4.12)$$

Inserting (4.12) into (4.11) yields the claim (4.6).

In order to prove (4.7), we first show

$$\int_{\mathbb{T}^2} |f(x+z) - f(x)| d^2x \leq \min \left\{ 2\|f\|_\infty, \frac{1}{2} \int_{\mathbb{T}^2} |\nabla f| d^2x \right\} \quad \text{for all } z \in \mathbb{R}^2. \quad (4.13)$$

Indeed, the upper bound of $2\|f\|_\infty$ in (4.13) is trivial. Furthermore, since f is periodic, it is sufficient to show the second upper bound in (4.13) only for $z \in (-\frac{1}{2}, \frac{1}{2})^2$. Thus the second bound in (4.13) follows from (4.8) with $p = 1$

$$\int_{\mathbb{T}^2} |f(x+z) - f(x)| d^2x \stackrel{(4.8)}{\leq} \int_{\mathbb{T}^2} |\nabla f(x) \cdot z| d^2x \leq \frac{1}{2} \int_{\mathbb{T}^2} |\nabla f(x)| d^2x$$

so that the proof of (4.13) is complete. With (4.13) at hand, estimate (4.7) now follows by direct integration

$$\begin{aligned} \int_{\mathbb{T}^2} \int_{\mathbb{R}^2 \setminus B_R} \frac{|f(x+z) - f(x)|^2}{|z|^3} d^2z d^2x &\leq 2\|f\|_{L^\infty} \int_{\mathbb{R}^2 \setminus B_R} \int_{\mathbb{T}^2} \frac{|f(x+z) - f(x)|}{|z|^3} d^2x d^2z \\ &\stackrel{(4.13)}{\leq} \frac{2\pi\|f\|_\infty}{R} \min \left\{ 4\|f\|_\infty, \int_{\mathbb{T}^2} |\nabla f| d^2x \right\}. \end{aligned} \quad (4.14)$$

It remains to prove (4.1), for which we use the real-space representation of the homogeneous $H^{1/2}$ -norm

$$\int_{\mathbb{T}^2} |\nabla^{1/2} f|^2 d^2x = \frac{1}{4\pi} \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \frac{|f(x+z) - f(x)|^2}{|z|^3} d^2z d^2x. \quad (4.15)$$

A proof of (4.15) is given in the appendix for completeness of the presentation. Without loss of generality, we may assume that f is not equal to a constant in \mathbb{T}^2 . Adding (4.5) – (4.7) to estimate the right hand side of (4.1), we therefore get

$$\begin{aligned} \int_{\mathbb{T}^2} |\nabla^{1/2} f|^2 d^2x &\leq \frac{r}{4} \int_{\mathbb{T}^2} |\nabla f|^2 d^2x \\ &\quad + \left(\frac{2}{\pi} \log \left(\frac{R}{r} \right) + \frac{1}{2R} \min \left\{ \frac{4\|f\|_\infty}{\int_{\mathbb{T}^2} |\nabla f| dx}, 1 \right\} \right) \|f\|_\infty \int_{\mathbb{T}^2} |\nabla f| d^2x. \end{aligned} \quad (4.16)$$

For $r = 2\varepsilon$ and $R = \max \left\{ 2\varepsilon, \min \left\{ \frac{4\|f\|_\infty}{\int_{\mathbb{T}^2} |\nabla f| dx}, 1 \right\} \right\}$ the claim (4.1) now follows from (4.16). \square

5 Proofs for the reduced energy F

In this section we give the proofs of the Theorems involving the reduced energy F . The proof of Theorem 3.5 is a direct consequence of Lemma 5.1 and Lemma 5.3. Similarly the proof of Theorem 3.6 follows immediately from Lemma 5.4 and Lemma 5.5. Finally, the proof of Theorem 3.7 is presented at the end of this section.

5.1 Proof of Theorem 3.5

Lemma 5.1 (Lower bound and compactness in the subcritical regime). *Let $\lambda < \lambda_c := \frac{\pi}{2}$ and $F_{\varepsilon,\lambda}$ as defined in (2.15). Then every sequence (m_ε) in $H^1(\mathbb{T}^2; \mathbb{S}^2)$ with*

$$\limsup_{\varepsilon \rightarrow 0} F_{\varepsilon,\lambda}[m_\varepsilon] < +\infty \quad (5.1)$$

converges in $L^1(\mathbb{T}^2; \mathbb{R}^3)$ (up to extracting a subsequence) towards a limit in $BV(\mathbb{T}^2; \{\pm e_3\})$. Furthermore, for every sequence (m_ε) in $L^1(\mathbb{T}^2; \mathbb{S}^2)$ with $m_\varepsilon \rightarrow m$ for some m in $L^1(\mathbb{T}^2; \mathbb{R}^3)$ we have

$$\liminf_{\varepsilon \rightarrow 0} F_{\varepsilon,\lambda}[m_\varepsilon] \geq \begin{cases} \left(1 - \frac{\lambda}{\lambda_c}\right) \int_{\mathbb{T}^2} |\nabla m_3| \, d^2x & \text{for } m \in BV(\mathbb{T}^2; \{\pm e_3\}), \\ +\infty & \text{otherwise.} \end{cases} \quad (5.2)$$

Proof of Lemma 5.1. We first show that for sufficiently small $\varepsilon > 0$ we have

$$F_{\varepsilon,\lambda}[m] \geq \left(1 - \frac{\lambda |\log c\varepsilon|}{\lambda_c |\log \varepsilon|}\right) \int_{\mathbb{T}^2} |\nabla m_3| \, d^2x \quad (5.3)$$

for all $m \in H^1(\mathbb{T}^2; \mathbb{S}^2)$, where $c > 0$ is a universal constant. Indeed, for $\lambda < \lambda_c$ we expect $\int_{\mathbb{T}^2} |\nabla m_3| \, dx$ to be small and hence it is sufficient to use Lemma 4.1 for m_3 in the weaker form (4.3). Recalling that $\|m_3\|_\infty \leq 1$ and $\lambda_c = \frac{\pi}{2}$, we get

$$\frac{\lambda}{|\log \varepsilon|} \int_{\mathbb{T}^2} |\nabla^{1/2} m_3|^2 \, d^2x \stackrel{(4.3)}{\leq} \frac{\lambda}{|\log \varepsilon|} \int_{\mathbb{T}^2} \frac{\varepsilon}{2} |\nabla m_3|^2 \, d^2x + \frac{\lambda \log(c_*/\varepsilon)}{\lambda_c |\log \varepsilon|} \int_{\mathbb{T}^2} |\nabla m_3| \, d^2x. \quad (5.4)$$

We also use the constraint $|m| = 1$ in the form of the well-known estimate

$$|\nabla m_3| \stackrel{(A.10)}{\leq} \frac{\varepsilon}{2} |\nabla m|^2 + \frac{1}{2\varepsilon} (1 - m_3^2). \quad (5.5)$$

which is obtained by differentiating $|m|^2 = 1$ and applying Young's inequality (see (A.10) in the Appendix for a proof). Now the claimed lower bound (5.3) follows from (5.4) and (5.5):

$$\begin{aligned}
F_{\varepsilon,\lambda}[m] &= \int_{\mathbb{T}^2} \left(\frac{\varepsilon}{2} |\nabla m|^2 + \frac{1}{2\varepsilon} (1 - m_3^2) \right) d^2x - \frac{\lambda}{|\log \varepsilon|} \int_{\mathbb{T}^2} |\nabla^{1/2} m_3|^2 d^2x \\
&\stackrel{(5.4)}{\geq} \left(1 - \frac{\lambda}{|\log \varepsilon|} \right) \int_{\mathbb{T}^2} \left(\frac{\varepsilon}{2} |\nabla m|^2 + \frac{1}{2\varepsilon} (1 - m_3^2) \right) d^2x \\
&\quad - \frac{\lambda \log(c_*/\varepsilon)}{\lambda_c |\log \varepsilon|} \int_{\mathbb{T}^2} |\nabla m_3| d^2x \\
&\stackrel{(5.5)}{\geq} \left(1 - \frac{\lambda \log(e^{\lambda_c} c_*/\varepsilon)}{\lambda_c |\log \varepsilon|} \right) \int_{\mathbb{T}^2} \left(\frac{\varepsilon}{2} |\nabla m|^2 + \frac{1}{2\varepsilon} (1 - m_3^2) \right) d^2x \\
&\stackrel{(5.5)}{\geq} \left(1 - \frac{\lambda \log(e^{\lambda_c} c_*/\varepsilon)}{\lambda_c |\log \varepsilon|} \right) \int_{\mathbb{T}^2} |\nabla m_3| d^2x.
\end{aligned} \tag{5.6}$$

Let m_ε be a sequence in $H^1(\mathbb{T}^2; \mathbb{S}^2)$ with bounded energy $\limsup_{\varepsilon \rightarrow 0} F_{\varepsilon,\lambda}[m_\varepsilon] < +\infty$. From the penultimate line in (5.6), $|m_\varepsilon| = 1$ and $\lambda < \lambda_c$ we obtain

$$0 = \limsup_{\varepsilon \rightarrow 0} \varepsilon F_{\varepsilon,\lambda}[m_\varepsilon] \stackrel{(5.6)}{\geq} \frac{1}{2} \left(1 - \frac{\lambda}{\lambda_c} \right) \limsup_{\varepsilon \rightarrow 0} \int_{\mathbb{T}^2} (m_{\varepsilon,1}^2 + m_{\varepsilon,2}^2) d^2x, \tag{5.7}$$

implying that the first two components $m_{\varepsilon,1}$ and $m_{\varepsilon,2}$ converge to zero in $L^2(\mathbb{T}^2)$ as $\varepsilon \rightarrow 0$. Moreover, (5.6) yields a uniform bound for $m_{\varepsilon,3}$ in BV , which by compactness of $BV(\mathbb{T}^2)$ in $L^1(\mathbb{T}^2)$ implies the existence of a convergent subsequence. Passing to another subsequence, we may assume that m_ε converges pointwise almost everywhere. Since $|m_\varepsilon| = 1$, we obtain $m = \pm e_3$ almost everywhere.

For the liminf inequality (5.2), we may assume without loss of generality that $\liminf_{\varepsilon \rightarrow 0} F_{\varepsilon,\lambda}[m_\varepsilon] < +\infty$. But then there is a subsequence (not relabelled) such that $\limsup_{\varepsilon \rightarrow 0} F_{\varepsilon,\lambda}[m_\varepsilon] < +\infty$ and by the compactness result and uniqueness of the limit we have $m \in BV(\mathbb{T}^2; \{\pm e_3\})$. Now the liminf inequality follows directly from (5.3), the fact that $\lim_{\varepsilon \rightarrow 0} \frac{\lambda |\log c\varepsilon|}{\lambda_c |\log \varepsilon|} = \frac{\lambda}{\lambda_c} < 1$ and lower semi-continuity of the BV -seminorm. \square

Before we begin with the construction of the upper bound, we define a family of asymptotically optimal profiles and record some of their properties (see Fig. 4).

Lemma 5.2 (Estimates for a family of asymptotically optimal profiles). *For $R \in (0, +\infty]$ and $\varepsilon > 0$, let $\xi_{\varepsilon,R} : \mathbb{R} \rightarrow [-1, 1]$ be the unique solution to the initial value problem*

$$\xi_{\varepsilon,R}(0) = 0 \quad \text{and} \quad \xi'_{\varepsilon,R} = \frac{1}{\varepsilon} (1 - \xi_{\varepsilon,R}^2)^{1/2} \left(1 - \xi_{\varepsilon,R}^2 + \left(\frac{\pi\varepsilon}{2R} \right)^2 \right)^{1/2}. \tag{5.8}$$

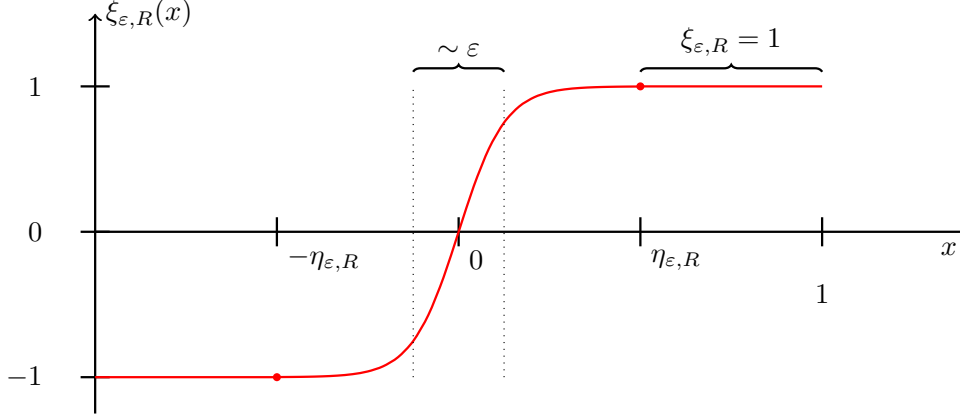


Figure 4: Family of asymptotically optimal profiles $\xi_{\varepsilon,R}$

Then $\xi_{\varepsilon,R}$ is non-decreasing and satisfies

$$\xi_{\varepsilon,R}(x) = -\xi_{\varepsilon,R}(-x) \quad \text{and} \quad |\xi_{\varepsilon,R}(x) - \text{sign}(x)| \leq 2e^{-2|x|/\varepsilon}. \quad (5.9)$$

Moreover, $\xi_{\varepsilon,R}(x) = 1$ if $x \geq \eta_{\varepsilon,R}$ and $\xi_{\varepsilon,R}(x) = -1$ if $x \leq -\eta_{\varepsilon,R}$, for some $\eta_{\varepsilon,R} \in (0, R]$. The contribution to the local part of the energy may be estimated as

$$\frac{1}{2} \int_{-\eta_{\varepsilon,R}}^{\eta_{\varepsilon,R}} \left(\frac{\varepsilon |\xi'_{\varepsilon,R}|^2}{1 - \xi_{\varepsilon,R}^2} + \frac{1 - \xi_{\varepsilon,R}^2}{\varepsilon} \right) dx \leq 2 + \frac{\pi^2 \varepsilon}{4R}. \quad (5.10)$$

Lastly, there is a universal constant $c > 0$ such that

$$\int_{-X}^X \int_{-X}^X \frac{|\xi_{\varepsilon,R}(x) - \xi_{\varepsilon,R}(y)|^2}{|x - y|^2} dx dy \geq 8 \log(cX/\varepsilon) \quad \text{for } X \geq 2\varepsilon. \quad (5.11)$$

Proof. The existence, uniqueness and monotonicity of $\xi_{\varepsilon,R}$ follows by direct integration. In particular, for $R < +\infty$, there exists a unique real number $\eta_{\varepsilon,R} > 0$, such that the solution of (5.8) satisfies $\xi_{\varepsilon,R}(s) \in (-1, 1)$ for $s \in (-\eta_{\varepsilon,R}, \eta_{\varepsilon,R})$ and $\xi_{\varepsilon,R}(\pm\eta_{\varepsilon,R}) = \pm 1$. For $R = +\infty$, we have $\xi_{\varepsilon,\infty} = \tanh(\cdot/\varepsilon)$ and the claim follows for $\eta_{\varepsilon,\infty} = +\infty$. Estimate (5.9) follows immediately from $\xi_{\varepsilon,\infty} \leq \xi_{\varepsilon,R} \leq 1$ for $x \geq 0$.

We will now show that $\eta_{\varepsilon,R} \leq R$ holds. Indeed, since $\xi_{\varepsilon,R}$ is strictly monotone on $(-\eta_{\varepsilon,R}, \eta_{\varepsilon,R})$, the inverse function theorem yields

$$\begin{aligned} \eta_{\varepsilon,R} &= \lim_{s \rightarrow 1^-} \xi_{\varepsilon,R}^{-1}(s) = \int_0^1 \left(\xi_{\varepsilon,R}^{-1} \right)'(s) ds \\ &= \int_0^1 \frac{\varepsilon}{\sqrt{(1-s^2) \left(1 - s^2 + \frac{\pi^2 \varepsilon^2}{4R^2} \right)}} ds \leq \int_0^1 \frac{2R}{\pi \sqrt{1-s^2}} ds = R. \end{aligned} \quad (5.12)$$

We turn to the proof of (5.10). By (5.8), we have

$$\begin{aligned} & \frac{\varepsilon |\xi'_{\varepsilon,R}|^2}{1 - \xi_{\varepsilon,R}^2} + \frac{1 - \xi_{\varepsilon,R}^2}{\varepsilon} \stackrel{(5.8)}{=} 2\xi'_{\varepsilon,R} + \frac{1}{\varepsilon} \left(\sqrt{1 - \xi_{\varepsilon,R}^2 + \left(\frac{\pi\varepsilon}{2R}\right)^2} - \sqrt{1 - \xi_{\varepsilon,R}^2} \right)^2 \\ & = 2\xi'_{\varepsilon,R} + \frac{1}{\varepsilon} \left(\int_0^{\frac{\pi\varepsilon}{2R}} \frac{s}{\sqrt{1 - \xi_{\varepsilon,R}^2 + s^2}} ds \right)^2 \leq 2\xi'_{\varepsilon,R} + \frac{\pi^2\varepsilon}{4R^2} \end{aligned} \quad (5.13)$$

and thus (5.10) follows from (5.13) by integration.

It remains to prove (5.11). By symmetry of $\xi_{\varepsilon,R}$ we have

$$\begin{aligned} & \int_{-X}^X \int_{\{\varepsilon \leq |z| \leq X\} \cap \{|x+z| \leq X\}} \frac{|\xi_{\varepsilon,R}(x+z) - \xi_{\varepsilon,R}(x)|^2}{|z|^2} dz dx \\ & = 2 \int_{-X}^0 \int_{\{\varepsilon \leq |z| \leq X\} \cap \{|x+z| \leq X\}} \frac{|\xi_{\varepsilon,R}(x+z) - \xi_{\varepsilon,R}(x)|^2}{|z|^2} dz dx \end{aligned} \quad (5.14)$$

As it turns out, it is sufficient to restrict the integral to a set where $|\xi_{\varepsilon,R}(x+z) - \xi_{\varepsilon,R}(x)| \gtrsim 1$ to obtain the correct leading order behavior

$$\begin{aligned} & \int_{-X}^0 \int_{\{\varepsilon \leq |z| \leq X\} \cap \{|x+z| \leq X\}} \frac{|\xi_{\varepsilon,R}(x+z) - \xi_{\varepsilon,R}(x)|^2}{|z|^2} dz dx \\ & \geq \int_{-X}^0 \int_{\varepsilon}^{x+X} \frac{|\xi_{\varepsilon,R}(y) - \xi_{\varepsilon,R}(x)|^2}{|y-x|^2} dy dx. \end{aligned}$$

Since $|1 - \xi_{\varepsilon,R}|$ decays exponentially with rate $1/\varepsilon$, we split the integral into the leading order and a lower order correction

$$\begin{aligned} & \int_{-X}^0 \int_{\varepsilon}^{x+X} \frac{|\xi_{\varepsilon,R}(y) - \xi_{\varepsilon,R}(x)|^2}{|y-x|^2} dy dx = \int_{-X}^0 \int_{\varepsilon}^{x+X} \frac{4}{|y-x|^2} dy dx \\ & \quad - \int_{-X}^0 \int_{\varepsilon}^{x+X} \frac{4 - |\xi_{\varepsilon,R}(y) - \xi_{\varepsilon,R}(x)|^2}{|y-x|^2} dy dx. \end{aligned} \quad (5.15)$$

The first term on the right hand side of (5.15) yields

$$\int_{-X}^0 \int_{\varepsilon}^{x+X} \frac{1}{|y-x|^2} dy dx = \log \left(\frac{\varepsilon + X}{\varepsilon} \right) - 1. \quad (5.16)$$

Thus, it is sufficient to show that the second term on the right hand side of (5.15) is bounded independently of ε . Indeed, using the exponential decay of $|1 - \xi_{\varepsilon,R}|$, we get

$$\int_{-X}^0 \int_{\varepsilon}^{x+X} \frac{4 - |\xi_{\varepsilon,R}(y) - \xi_{\varepsilon,R}(x)|^2}{|y-x|^2} dy dx \lesssim \int_0^{\infty} \int_1^{\infty} \frac{e^{-2x} + e^{-2y}}{|x+y|^2} dx dy \lesssim 1. \quad (5.17)$$

Together, (5.14) – (5.17) yield the claim (5.11). \square

For the special case $\lambda = 0$, the Γ -convergence and in particular the construction of a recovery sequence is a classical result, relying on the optimal one-dimensional transition profiles to smooth out the jump discontinuity in the limit configuration [3]. As it turns out, this construction also works for $\lambda > 0$, where $F_{\varepsilon,\lambda}$ is nonlocal. We will use a construction based on the nearly optimal profile $\xi_{\varepsilon,R}$ from Lemma 5.2. As the calculations for the local part of the energy are well-known, our focus is on the contribution of the homogeneous $H^{1/2}$ -norm. Recall that we need to prove a lower bound for the $H^{1/2}$ -norm in order to obtain an upper bound for F .

Lemma 5.3 (Construction of a recovery sequence in the subcritical and critical regime). *Let $\lambda \leq \lambda_c$ and $m \in L^1(\mathbb{T}^2; \mathbb{S}^2)$. Then there is a sequence (m_ε) in $H^1(\mathbb{T}^2; \mathbb{S}^2)$ with*

$$\limsup_{\varepsilon \rightarrow 0} F_{\varepsilon,\lambda}[m_\varepsilon] \leq F_{*,\lambda}[m], \quad (5.18)$$

where $F_{\varepsilon,\lambda}$ is given by (2.15), and $F_{*,\lambda}$ is given by (3.25) for $\lambda < \lambda_c$ or (3.33) for $\lambda = \lambda_c$, respectively.

Proof of Lemma 5.3. It is sufficient to prove the limsup inequality under the additional assumption that $m = (\chi_A - \chi_{\mathbb{T}^2 \setminus A})e_3$ for a set $A \subset \mathbb{T}^2$ with smooth boundary. By standard density results (see e.g. [47, Prop. 12.20]) and a diagonal argument, the limsup inequality then extends to arbitrary $A \subset \mathbb{T}^2$ with finite perimeter for $\lambda < \lambda_c$ or to measurable $A \subset \mathbb{T}^2$ for the $\lambda = \lambda_c$ case. Since $F_{*,\lambda}[m] = +\infty$ for $m \notin BV(\mathbb{T}^2, \{\pm e_3\})$ when $\lambda < \lambda_c$ or for $m \notin L^1(\mathbb{T}^2, \{\pm e_3\})$ when $\lambda = \lambda_c$, this yields the claim.

Our strategy is to adapt the optimal profiles $\xi_{\varepsilon,R}$ from Lemma 5.2 to the two-dimensional setting by means of the signed distance function d , given by $d(x) := \text{dist}(x, A^c) - \text{dist}(x, A)$. Without loss of generality, we may assume $0 < |A| < 1$ (otherwise take $m_\varepsilon \equiv \pm e_3$). To fix the notation, let $\nu : \partial A \rightarrow \mathbb{R}^2$ denote the smooth inward normal to A and $\tau : \partial A \rightarrow \mathbb{R}^2$, $\tau = \nu^\perp$ denote a smooth tangent vector field to ∂A obtained by a counter-clockwise 90° rotation of ν . As ∂A is assumed to be smooth, there exists a tubular neighborhood $(\partial A)_R = \bigcup_{x \in \partial A} B_R(x) \subset \mathbb{T}^2$ for some $R > 0$ such that the projection $p : (\partial A)_R \rightarrow \partial A$, $p(x) := \text{argmin}_{y \in \partial A} |x - y|$ is single-valued and hence well-defined. Furthermore, the projection p and the signed distance function d are smooth on $(\partial A)_R$ and the identity

$$x = p(x) + d(x)\nu(p(x)) \quad (5.19)$$

holds for all $x \in (\partial A)_R$, see e.g. [26, Lemma 14.16].

With the necessary notation at hand, we define the recovery sequence by

$$m_\varepsilon(x) = \xi_{\varepsilon,R}(d(x))e_3 + \sqrt{1 - \xi_{\varepsilon,R}^2(d(x))} \tau(p(x)). \quad (5.20)$$

Recall that $\eta_{\varepsilon,R} \leq R$, (see (5.12)) and hence the function m_ε is Lipschitz continuous and piecewise smooth.

It is easy to see that $m_\varepsilon \rightarrow m$ in $L^1(\mathbb{T}^2)$, and for the sake of completeness, we briefly mention how to compute the contribution of the local energy terms. Since $\tau \perp e_3$, $(\tau \circ p) \cdot \nabla(\tau \circ p) = 0$ and $|\nabla d| = 1$ almost everywhere, the squared gradient of m_ε can be estimated by

$$|\nabla m_\varepsilon|^2 = \frac{|\xi'_{\varepsilon,R}(d)|^2}{1 - \xi_{\varepsilon,R}^2(d)} + (1 - \xi_{\varepsilon,R}^2(d)) |\nabla(\tau \circ p)|^2 \leq \frac{|\xi'_{\varepsilon,R}(d)|^2}{1 - \xi_{\varepsilon,R}^2(d)} + C_A, \quad (5.21)$$

where $C_A > 0$ is a constant that depends only on A for all $R \leq R_A$, where $R_A > 0$ depends only on A . In the following, C_A may change from line to line.

We next employ the co-area formula, to reduce to the one-dimensional case:

$$\begin{aligned} & \int_{\mathbb{T}^2} \left(\frac{\varepsilon}{2} |\nabla m_\varepsilon|^2 + \frac{1}{2\varepsilon} (1 - m_{\varepsilon,3}^2) \right) d^2x \\ & \stackrel{(5.21)}{\leq} \int_{(\partial A)_{\eta_{\varepsilon,R}}} \left(\frac{\varepsilon |\xi'_{\varepsilon,R}(d)|^2}{2(1 - \xi_{\varepsilon,R}^2(d))} + \frac{1}{2\varepsilon} (1 - \xi_{\varepsilon,R}^2(d)) \right) d^2x + \varepsilon C_A \\ & \leq \int_{-\eta_{\varepsilon,R}}^{\eta_{\varepsilon,R}} \left(\frac{\varepsilon |\xi'_{\varepsilon,R}(s)|^2}{2(1 - \xi_{\varepsilon,R}^2(s))} + \frac{1}{2\varepsilon} (1 - \xi_{\varepsilon,R}^2(s)) \right) \mathcal{H}^1(\{d(x) = s\}) ds + \varepsilon C_A. \end{aligned} \quad (5.22)$$

Inserting the estimate for the one-dimensional profile from Lemma 5.2, we obtain

$$\begin{aligned} & \int_{-\eta_{\varepsilon,R}}^{\eta_{\varepsilon,R}} \left(\frac{\varepsilon |\xi'_{\varepsilon,R}(s)|^2}{2(1 - \xi_{\varepsilon,R}^2(s))} + \frac{1}{2\varepsilon} (1 - \xi_{\varepsilon,R}^2(s)) \right) \mathcal{H}^1(\{d(x) = s\}) ds \\ & \stackrel{(5.10)}{\leq} \sup_{-\eta_{\varepsilon,R} \leq s \leq \eta_{\varepsilon,R}} \mathcal{H}^1(\{d(x) = s\}) \left(2 + O\left(\frac{\varepsilon}{R}\right) \right). \end{aligned} \quad (5.23)$$

Since ∂A and the signed distance function d are smooth in $(\partial A)_R$, we have

$$\lim_{s \rightarrow 0} \mathcal{H}^1(\{d(x) = s\}) = \mathcal{H}^1(\partial A). \quad (5.24)$$

In the limit $\varepsilon \rightarrow 0$, then $R \rightarrow 0$, estimates (5.22), (5.23) and (5.12) hence imply

$$\limsup_{R \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \int_{\mathbb{T}^2} \left(\frac{\varepsilon}{2} |\nabla m_\varepsilon|^2 + \frac{1}{2\varepsilon} (1 - m_{\varepsilon,3}^2) \right) \leq 2\mathcal{H}^1(\partial A). \quad (5.25)$$

We now turn to the estimate of the nonlocal term in the energy F . As for the local terms, our strategy is to use the one-dimensional estimates from Lemma 5.2. Invoking the coarea formula twice and inserting (5.20), we get

$$\begin{aligned} & \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \frac{|m_{\varepsilon,3}(x) - m_{\varepsilon,3}(y)|^2}{|x - y|^3} d^2x d^2y \\ & \geq \int_{-R}^R \int_{\{x: d(x) = \rho'\}} \left(\int_{-R}^R \int_{\{y: d(y) = \rho\}} \frac{|\xi_{\varepsilon,R}(\rho') - \xi_{\varepsilon,R}(\rho)|^2}{|x - y|^3} d\mathcal{H}^1(y) d\rho \right) d\mathcal{H}^1(x) d\rho'. \end{aligned} \quad (5.26)$$

We claim that the integrals over curves tangential to the boundary may be estimated as follows: For every $\delta > 0$, there is an $R_{\delta,A}$ such that

$$\int_{\{x: d(x)=\rho'\}} \int_{\{y: d(y)=\rho\}} \frac{1}{|x-y|^3} d\mathcal{H}^1(y) d\mathcal{H}^1(x) \geq (1-\delta) \frac{2\mathcal{H}^1(\partial A)}{(\rho-\rho')^2}, \quad (5.27)$$

for all $R \leq R_{\delta,A}$ and all $\rho \neq \rho' \in (-R, R)$. Assuming for a moment that (5.27) holds, we conclude by inserting (5.27) into (5.26) and applying the one-dimensional estimate (5.11)

$$\begin{aligned} \frac{\lambda}{|\log \varepsilon|} \int_{\mathbb{T}^2} |\nabla^{1/2} m_{\varepsilon,3}|^2 d^2x &\stackrel{(5.26),(5.27)}{\geq} (1-\delta) \frac{\lambda \mathcal{H}^1(\partial A)}{2\pi |\log \varepsilon|} \int_{-R}^R \int_{-R}^R \frac{|\xi_{\varepsilon,R}(\rho) - \xi_{\varepsilon,R}(\rho')|^2}{|\rho - \rho'|^2} d\rho' d\rho \\ &\stackrel{(5.11)}{\geq} (1-\delta) 2\mathcal{H}^1(\partial A) \frac{\lambda}{\lambda_c} \frac{\log(cR/\varepsilon)}{|\log \varepsilon|}. \end{aligned}$$

Since δ was arbitrary, we obtain

$$\liminf_{R \rightarrow 0} \liminf_{\varepsilon \rightarrow 0} \frac{\lambda}{|\log \varepsilon|} \int_{\mathbb{T}^2} |\nabla^{1/2} m_{\varepsilon,3}|^2 d^2x \geq 2\mathcal{H}^1(\partial A) \frac{\lambda}{\lambda_c}. \quad (5.28)$$

Together, (5.25) and (5.28) imply the limsup inequality by a standard diagonal argument.

It remains to prove (5.27), for which we fix $x \in (\partial A)_R$ with $d(x) = \rho'$ and pass to curvilinear coordinates in a neighborhood of $\tilde{x} := p(x) \in \partial A$. More precisely, let the curve $\gamma : (-R^{1/2}, R^{1/2}) \rightarrow \partial A$ be a parametrization by arclength of a neighborhood of \tilde{x} in ∂A with $\gamma(0) = \tilde{x}$. Then, for all $R \leq R_A$ with some $R_A > 0$ the function

$$\Psi(\sigma, \rho) := \gamma(\sigma) + \nu(\gamma(\sigma))\rho \quad (5.29)$$

is a diffeomorphism from $(-R^{1/2}, R^{1/2}) \times (-R, R)$ onto its image, which we denote by $\Gamma_{\tilde{x}}$. The choice $R^{1/2}$ will become clear later. Note that due to compactness of ∂A , we may choose R_A independent of \tilde{x} . A transformation of variables then yields

$$\int_{\{y: d(y)=\rho\} \cap \Gamma_{p(x)}} \frac{1}{|x-y|^3} d\mathcal{H}^1(y) = \int_{-R^{1/2}}^{R^{1/2}} \frac{(1 + \kappa(\gamma(\sigma))\rho)}{|\Psi(0, \rho') - \Psi(\sigma, \rho)|^3} d\sigma, \quad (5.30)$$

where $\kappa(\tilde{y})$ denotes the signed curvature of ∂A at \tilde{y} (negative if A is convex). Since the curvature of ∂A is bounded, there is, for any $\delta > 0$, an $R_{\delta,A} > 0$ such that for all $R \leq R_{\delta,A}$ we have

$$|\kappa| R \leq \delta \quad \text{and} \quad |\Psi(0, \rho') - \Psi(\sigma, \rho)| \leq (1 + \delta) \sqrt{\sigma^2 + (\rho - \rho')^2}. \quad (5.31)$$

We conclude that, for any $\tilde{\delta} > 0$, there is an $\tilde{R}_{\tilde{\delta}, A} > 0$ such that for all $R \leq \tilde{R}_{\tilde{\delta}, A}$ and all $\rho, \rho' \in (-R, R)$ we have

$$\begin{aligned} \int_{\{y: d(y)=\rho\} \cap \Gamma_{p(x)}} \frac{1}{|x-y|^3} d\mathcal{H}^1(y) &\stackrel{(5.30), (5.31)}{\geq} (1-\tilde{\delta}) \int_{-R^{1/2}}^{R^{1/2}} \frac{1}{(\sigma^2 + (\rho - \rho'))^{3/2}} d\sigma \\ &= (1-\tilde{\delta}) \frac{2}{(\rho - \rho')^2} \frac{R^{1/2}}{\sqrt{R + (\rho - \rho')^2}} \geq (1-2\tilde{\delta}) \frac{2}{(\rho - \rho')^2}. \end{aligned} \quad (5.32)$$

Integrating (5.32) over x and invoking (5.24) we obtain (5.27). \square

5.2 Proof of Theorem 3.6

We begin with the proof of the lower bound in Theorem 3.6, which is the subject of Lemma 5.4. The proof of Theorem 3.6 is completed with the construction of the upper bound, carried out in Lemma 5.5.

Lemma 5.4. *Let $\lambda_c := \frac{\pi}{2}$ and $F_{\varepsilon, \lambda}$ as defined in (2.15). Then there is a universal constant $\delta > 0$ such that for all $\varepsilon < 1/2$ and all*

$$\lambda_c \leq \lambda < \delta |\log \varepsilon| \quad (5.33)$$

the family of functionals $\{F_{\varepsilon, \lambda}\}$ is bounded below by

$$\min F_{\varepsilon, \lambda} \gtrsim -\frac{\lambda \varepsilon^{\frac{\lambda_c - \lambda}{\lambda}}}{|\log \varepsilon|}. \quad (5.34)$$

Moreover, the profiles achieving the optimal scaling can be characterized as follows: For any $\gamma > 0$ and all $m \in H^1(\mathbb{T}^2; \mathbb{S}^2)$ which satisfy

$$F_{\varepsilon, \lambda}[m] \leq -\frac{\lambda \varepsilon^{\frac{\lambda_c - \lambda}{\lambda}}}{|\log \varepsilon|} \gamma, \quad (5.35)$$

there holds

$$\int_{\mathbb{T}^2} |\nabla m_3| dx \leq \int_{\mathbb{T}^2} \left(\frac{\varepsilon}{2} |\nabla m|^2 + \frac{1 - m_{\varepsilon, 3}^2}{2\varepsilon} \right) dx \leq \frac{\lambda}{|\log \varepsilon|} \int_{\mathbb{T}^2} |\nabla^{1/2} m_3|^2 dx, \quad (5.36)$$

and the above quantities agree to leading order and scale like $\varepsilon^{\frac{\lambda_c - \lambda}{\lambda}}$, i.e. if A and B are any of the three quantities in (5.36), we have

$$A \sim \varepsilon^{\frac{\lambda_c - \lambda}{\lambda}} \quad \text{and} \quad |A - B| \lesssim \frac{\lambda}{|\log \varepsilon|} A, \quad (5.37)$$

where the constants may depend on γ .

Proof. By (4.1), we may bound the energy from below by

$$F_{\varepsilon,\lambda}[m] \stackrel{(4.1)}{\geq} \left(1 - \frac{\lambda}{|\log \varepsilon|}\right) \int_{\mathbb{T}^2} \frac{\varepsilon}{2} |\nabla m|^2 + \frac{1}{2\varepsilon} (1 - m_3^2) dx - \frac{\lambda \log \left(c_* \max \left\{ 1, \min \left\{ \frac{1}{\varepsilon \int_{\mathbb{T}^2} |\nabla m_3| dx}, \frac{1}{\varepsilon} \right\} \right\} \right)}{\lambda_c |\log \varepsilon|} \int_{\mathbb{T}^2} |\nabla m_3| dx. \quad (5.38)$$

Without loss of generality, we may assume that $\int_{\mathbb{T}^2} |\nabla m_3| dx > 0$. We first consider the case $\min \left\{ \frac{1}{\varepsilon \int_{\mathbb{T}^2} |\nabla m_3| dx}, \frac{1}{\varepsilon} \right\} \leq 1$, for which, with the help of (A.10), the estimate in (5.38) turns into

$$F_{\varepsilon,\lambda}[m] \geq \left(1 - \frac{\lambda \log(c_*^{1/\lambda_c})}{|\log \varepsilon|}\right) \int_{\mathbb{T}^2} |\nabla m_3| dx \stackrel{(5.33)}{\geq} (1 - C\delta) \int_{\mathbb{T}^2} |\nabla m_3| dx \quad (5.39)$$

for some universal constant $C > 0$. For $\delta < 1/C$, the right hand side of (5.39) is positive and the lower bound follows. Hence, we may assume $\min \left\{ \frac{1}{\varepsilon \int_{\mathbb{T}^2} |\nabla m_3| dx}, \frac{1}{\varepsilon} \right\} > 1$ so that (5.38) implies

$$F_{\varepsilon,\lambda}[m] \geq \left(1 - \frac{\lambda}{|\log \varepsilon|}\right) \int_{\mathbb{T}^2} \frac{\varepsilon}{2} |\nabla m|^2 + \frac{1}{2\varepsilon} (1 - m_3^2) dx - \frac{\lambda \log \left(\frac{c_*}{\varepsilon \int_{\mathbb{T}^2} |\nabla m_3| dx} \right)}{\lambda_c |\log \varepsilon|} \int_{\mathbb{T}^2} |\nabla m_3| dx. \quad (5.40)$$

Abbreviating the energetic cost for m to deviate from the optimal Bloch wall profile by

$$D_\varepsilon[m] := \int_{\mathbb{T}^2} \frac{\varepsilon}{2} |\nabla m|^2 + \frac{1}{2\varepsilon} (1 - m_3^2) dx - \int_{\mathbb{T}^2} |\nabla m_3| dx, \quad (5.41)$$

and inserting $\mu := \varepsilon^{\frac{\lambda - \lambda_c}{\lambda}} \int_{\mathbb{T}^2} |\nabla m_3| dx$ and $c_{**} := c_* e^{\lambda_c}$ into the lower bound in (5.40), we get

$$F_{\varepsilon,\lambda}[m] \geq \left(1 - \frac{\lambda}{|\log \varepsilon|}\right) D_\varepsilon[m] - \frac{\lambda \log \left(\frac{c_{**}}{\mu} \right)}{\lambda_c |\log \varepsilon|} \mu \varepsilon^{\frac{\lambda_c - \lambda}{\lambda}}. \quad (5.42)$$

Since $\sup_{\mu > 0} \mu \log(c_{**}/\mu) = c_{**}/e$, and since $D_\varepsilon[m] \geq 0$ by (A.10), the lower bound in (5.34) follows.

We now turn to the proof of (5.37). Note that (A.10) and $F_{\varepsilon,\lambda}[m] \leq 0$ yield

$$\int_{\mathbb{T}^2} |\nabla m_3| dx \leq \int_{\mathbb{T}^2} \left(\frac{\varepsilon}{2} |\nabla m|^2 + \frac{1 - m_3^2}{2\varepsilon} \right) dx \leq \frac{\lambda}{|\log \varepsilon|} \int_{\mathbb{T}^2} |\nabla^{1/2} m_3| dx. \quad (5.43)$$

For (5.37) it is hence sufficient to show

$$\int_{\mathbb{T}^2} |\nabla m_3| dx \sim \varepsilon^{\frac{\lambda_c - \lambda}{\lambda}} \quad \text{and} \quad \frac{\lambda}{|\log \varepsilon|} \int_{\mathbb{T}^2} |\nabla^{1/2} m_3| dx - \int_{\mathbb{T}^2} |\nabla m_3| dx \lesssim \frac{\lambda \varepsilon^{\frac{\lambda_c - \lambda}{\lambda}}}{|\log \varepsilon|}, \quad (5.44)$$

where here and in the rest of the proof the constants may depend on γ . We combine the lower bound for the energy (5.42) with the upper bound (5.35) to obtain $\mu \log(c_{**}/\mu) \gtrsim 1$, which in turn implies $\mu \sim 1$. Hence, the first item in (5.44) may be estimated as

$$\int_{\mathbb{T}^2} |\nabla m_3| dx = \mu \varepsilon^{\frac{\lambda_c - \lambda}{\lambda}} \sim \varepsilon^{\frac{\lambda_c - \lambda}{\lambda}}.$$

For $\delta > 0$ sufficiently small universal and $\mu \sim 1$, the second item in (5.44) follows from (5.42):

$$\frac{\lambda}{|\log \varepsilon|} \int_{\mathbb{T}^2} |\nabla^{1/2} m_3| dx - \int_{\mathbb{T}^2} |\nabla m_3| dx = -F_{\varepsilon, \lambda}[m] + D_\varepsilon[m] \stackrel{(5.42)}{\lesssim} \frac{\lambda \varepsilon^{\frac{\lambda_c - \lambda}{\lambda}}}{|\log \varepsilon|}. \quad (5.45)$$

This concludes the proof. \square

Lemma 5.5 (Upper bound in the supercritical regime). *There is a constant $0 < K < 1$ such that for every (ε, λ) with*

$$\lambda_c < \lambda \quad \text{and} \quad 0 < \varepsilon^{\frac{\lambda - \lambda_c}{\lambda}} < K, \quad (5.46)$$

there is $m_{\varepsilon, \lambda} \in H^1(\mathbb{T}^2; \mathbb{S}^2)$ which satisfies

$$F_{\varepsilon, \lambda}[m_{\varepsilon, \lambda}] \lesssim -\frac{\lambda \varepsilon^{\frac{\lambda_c - \lambda}{\lambda}}}{|\log \varepsilon|}. \quad (5.47)$$

Proof. We make an ansatz with N transitions equally separated by $1/N$ -sized regions of approximately constant magnetization. More precisely, we take the transitions as solutions of the optimal profile ODE and define

$$m_{\varepsilon, N}(x_1, x_2) = \begin{cases} \xi_{\varepsilon, \infty} \left(\frac{x_1 - \frac{1}{2N}}{\varepsilon} \right) e_3 + \sqrt{1 - \xi_{\varepsilon, \infty}^2 \left(\frac{x_1 - \frac{1}{2N}}{\varepsilon} \right)} e_2, & \text{for } x_1 \in \left[0, \frac{1}{N}\right] \\ \xi_{\varepsilon, \infty} \left(\frac{\frac{3}{2N} - x_1}{\varepsilon} \right) e_3 + \sqrt{1 - \xi_{\varepsilon, \infty}^2 \left(\frac{\frac{3}{2N} - x_1}{\varepsilon} \right)} e_2, & \text{for } x_1 \in \left[\frac{1}{N}, \frac{2}{N}\right] \end{cases} \quad (5.48)$$

extended periodically to \mathbb{T}^2 (see Fig. 2). Applying Lemma 5.2 with $X = \frac{1}{2N}$ and using symmetries of $m_{\varepsilon, N}$, we get

$$\int_{\mathbb{T}^2} \left(\frac{\varepsilon}{2} |\nabla m_{\varepsilon, N}|^2 + \frac{1 - m_{(\varepsilon, N), 3}^2}{2\varepsilon} \right) dx \leq 2N \quad (5.49)$$

and, for all $\varepsilon < \frac{1}{4N}$, we have

$$\begin{aligned}
& \int_{\mathbb{T}^2} |\nabla^{1/2} m_{(\varepsilon, N), 3}|^2 d^2x \stackrel{(A.14)}{=} \frac{1}{4\pi} \int_{\mathbb{T}} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|m_{(\varepsilon, N), 3}(x_1) - m_{(\varepsilon, N), 3}(y_1)|^2}{(|x_1 - y_1|^2 + s^2)^{3/2}} ds dx_1 dy_1 \\
& \geq \frac{1}{2\pi} \sum_{k=1}^N \int_{\frac{k-1}{N}}^{\frac{k}{N}} \int_{\frac{k-1}{N}}^{\frac{k}{N}} \frac{|m_{(\varepsilon, N), 3}(x_1) - m_{(\varepsilon, N), 3}(y_1)|^2}{|x_1 - y_1|^2} dx_1 dy_1 \\
& = \frac{N}{4\lambda_c} \int_{-\frac{1}{2N}}^{\frac{1}{2N}} \int_{-\frac{1}{2N}}^{\frac{1}{2N}} \frac{|\xi_{\varepsilon, \infty}(x) - \xi_{\varepsilon, \infty}(y)|^2}{|x - y|^2} dx dy \stackrel{(5.11)}{\geq} 2N \frac{\log(\frac{c}{2\varepsilon N})}{\lambda_c}.
\end{aligned} \tag{5.50}$$

To obtain the upper bound, we combine estimates (5.49) and (5.50) and optimize in $N \in \mathbb{N}$. The choice $N := 2 \left\lfloor K\varepsilon^{\frac{\lambda_c - \lambda}{\lambda}} \right\rfloor$ is admissible because $N \stackrel{(5.46)}{\geq} 2$ and $\varepsilon N \leq 2K \leq \frac{1}{4}$ for $K \leq \frac{1}{8} \min\{1, c\}$. Since $0 < \varepsilon < 1$, we get

$$F_{\varepsilon, \lambda}[m_{\varepsilon, N}] \leq 2N \left(1 - \frac{\lambda \log(\frac{c}{2\varepsilon N})}{\lambda_c |\log \varepsilon|} \right) \leq -\frac{C\lambda\varepsilon^{\frac{\lambda_c - \lambda}{\lambda}}}{|\log \varepsilon|}, \tag{5.51}$$

for some universal $C > 0$, which is the desired estimate. \square

5.3 Proof of Theorem 3.7

Proof of Theorem 3.7. We start by proving item (i). Inserting (3.31) into the lower bound (5.3), we get for sufficiently small $\varepsilon > 0$

$$\begin{aligned}
F_{\varepsilon, \lambda}[m] & \geq \left(1 - \frac{\log(\varepsilon c) \log(\varepsilon/\beta_1)}{\log(\varepsilon)^2} \right) \int_{\mathbb{T}^2} |\nabla m_3| d^2x \\
& \geq \left(\frac{|\log(\varepsilon)| \log(c/\beta_1) + \log(c) \log(\beta_1)}{|\log(\varepsilon)|^2} \right) \int_{\mathbb{T}^2} |\nabla m_3| d^2x.
\end{aligned} \tag{5.52}$$

For $\beta_1 < c$, the bracket is positive, which shows that the minimal value of $\min F_{\varepsilon, \lambda} = 0$ is only attained for $m \equiv \pm e_3$. Since $\varepsilon^{\frac{\lambda_+(\varepsilon) - \lambda_c}{\lambda_+(\varepsilon)}} \leq \frac{2}{\beta_2}$ for sufficiently small $\varepsilon > 0$, the second part follows from Lemma 5.5.

To proceed, we next establish the estimate

$$\int_{\mathbb{T}^2} |\nabla m_3| d^2x \lesssim \max\{1, |\log \varepsilon| F_{\varepsilon, \lambda_c}[m]\}. \tag{5.53}$$

It is enough to show that there are constants $C, \varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have

$$\int_{\mathbb{T}^2} |\nabla m_3| d^2x \geq C \implies F_{\varepsilon, \lambda_c}[m] \gtrsim \frac{1}{|\log \varepsilon|} \int_{\mathbb{T}^2} |\nabla m_3| d^2x. \tag{5.54}$$

Indeed, by (4.1), we may bound the energy from below by

$$F_{\varepsilon, \lambda_c}[m_\varepsilon] \stackrel{(4.1)}{\geq} \left(1 - \frac{\lambda_c}{|\log \varepsilon|}\right) \int_{\mathbb{T}^2} \left(\frac{\varepsilon}{2} |\nabla m_\varepsilon|^2 + \frac{1}{2\varepsilon} (1 - m_{\varepsilon,3}^2)\right) dx - \frac{\log\left(c_* \max\left\{1, \min\left\{\frac{1}{\varepsilon \int_{\mathbb{T}^2} |\nabla m_{\varepsilon,3}| dx}, \frac{1}{\varepsilon}\right\}\right\}\right)}{|\log \varepsilon|} \int_{\mathbb{T}^2} |\nabla m_{\varepsilon,3}| dx. \quad (5.55)$$

We first consider the case $\min\left\{\frac{1}{\varepsilon \int_{\mathbb{T}^2} |\nabla m_{\varepsilon,3}| dx}, \frac{1}{\varepsilon}\right\} \leq 1$, for which (5.55) turns into

$$F_{\varepsilon, \lambda_c}[m] \geq \left(1 - \frac{\lambda_c + \log(c_*)}{|\log \varepsilon|}\right) \int_{\mathbb{T}^2} |\nabla m_{\varepsilon,3}| dx \gtrsim \int_{\mathbb{T}^2} |\nabla m_3| dx. \quad (5.56)$$

For the remaining case, we have $\min\left\{\frac{1}{\varepsilon \int_{\mathbb{T}^2} |\nabla m_3| dx}, \frac{1}{\varepsilon}\right\} \geq 1$ and (5.55) implies

$$F_{\varepsilon, \lambda_c}[m_\varepsilon] \geq \left(1 - \frac{\lambda_c}{|\log \varepsilon|}\right) \int_{\mathbb{T}^2} \left(\frac{\varepsilon}{2} |\nabla m_\varepsilon|^2 + \frac{1}{2\varepsilon} (1 - m_{\varepsilon,3}^2)\right) dx - \frac{\log\left(\frac{c_*}{\varepsilon \int_{\mathbb{T}^2} |\nabla m_{\varepsilon,3}| dx}\right)}{|\log \varepsilon|} \int_{\mathbb{T}^2} |\nabla m_{\varepsilon,3}| dx \stackrel{(A.10)}{\geq} - \frac{\log\left(\frac{c_{**}}{\int_{\mathbb{T}^2} |\nabla m_3|^2 dx}\right)}{|\log \varepsilon|} \int_{\mathbb{T}^2} |\nabla m_3|^2 dx, \quad (5.57)$$

where we have inserted $c_{**} := c_* e^{\lambda_c}$. The estimate (5.54) follows with the choice $C = 2c_{**}$.

With (5.53) at hand, we now prove item (ii), starting with the lower bound. Let $m_\varepsilon \rightarrow m$ in $L^1(\mathbb{T}^2)$ for some $m \in L^1(\mathbb{T}^2; \mathbb{R}^3)$. Lemma 5.4 yields

$$\liminf_{\varepsilon \rightarrow 0} F_{\varepsilon, \lambda_c}[m_\varepsilon] \geq 0, \quad (5.58)$$

which proves the lower bound in case that $m \in L^1(\mathbb{T}^2; \{\pm e_3\})$. For the remaining case, we may assume $\int_{\mathbb{T}^2} (1 - m_{\varepsilon,3}^2) d^2x \gtrsim 1$. For sufficiently small ε , estimates (4.1) and (5.53) then yield

$$\int_{\mathbb{T}^2} (1 - m_{\varepsilon,3}^2) d^2x \lesssim \varepsilon \left(F_{\varepsilon, \lambda}[m_\varepsilon] + \frac{\lambda_c}{|\log \varepsilon|} \int_{\mathbb{T}^2} |\nabla^{1/2} m_{\varepsilon,3}|^2 d^2x\right) \stackrel{(4.1)}{\lesssim} \varepsilon \left(F_{\varepsilon, \lambda}[m_\varepsilon] + \int_{\mathbb{T}^2} |\nabla m_{\varepsilon,3}| d^2x\right) \stackrel{(5.53)}{\lesssim} \varepsilon (1 + |\log \varepsilon| F_{\varepsilon, \lambda_c}[m_\varepsilon]), \quad (5.59)$$

which implies $\liminf_{\varepsilon \rightarrow 0} F_{\varepsilon, \lambda_c}[m_\varepsilon] = +\infty$ for $m \in L^1(\mathbb{T}^2; \mathbb{R}^3) \setminus L^1(\mathbb{T}^2; \{\pm e_3\})$. Since the construction of the upper bound was already carried out in Lemma 5.3, the proof is complete.

To prove item (iii), we again make use of the construction in Lemma 5.5. However, this time we take $N = \lfloor \log(|\log \varepsilon|) \rfloor$. Analogous to (5.51), we get for sufficiently small ε

$$F_{\varepsilon, \lambda}[m_{\varepsilon, N}] \leq 2N \left(1 - \frac{\log(\frac{2\varepsilon N}{c})}{\log \varepsilon} \right) \lesssim \frac{N \log N}{|\log \varepsilon|} \longrightarrow 0, \quad \text{for } \varepsilon \rightarrow 0. \quad (5.60)$$

Therefore, it remains to show that $m_{\varepsilon, N}$ is not compact in the strong L^1 -topology. Since $\int_{\mathbb{T}^2} |m_{\varepsilon, N}|^2 \, d^2x = 1$, any possible limit \tilde{m} of (a subsequence of) $m_{\varepsilon, N}$ in the strong topology needs to satisfy $\int_{\mathbb{T}^2} |\tilde{m}|^2 \, d^2x = 1$. However, since $\varepsilon N \rightarrow 0$ as $\varepsilon \rightarrow 0$, it is clear that $m_{\varepsilon, N}$ converges weakly to zero in $L^2(\mathbb{T}^2)$, leading to a contradiction.

Finally, item (iv) follows directly from (5.59), (5.53) and the compact embedding $BV(\mathbb{T}^2) \hookrightarrow L^1(\mathbb{T}^2)$. \square

6 Stray field estimates and reduction of the full energy

The goal of this section is to make the heuristic reduction in section 2.1 rigorous. We prove the following

Lemma 6.1 (Reduction of the energy). *There is a universal constant $C > 0$ such that energy E is bounded below by*

$$\begin{aligned} E[m] \geq \ell^2 t + (1 - Ct^2) \int_{\mathbb{T}_\ell^2 \times (0, t)} |\nabla m|^2 + (Q - 1)(m_1^2 + m_2^2) \, d^3x \\ - 2 \int_{\mathbb{T}_\ell^2 \times (0, t)} m_3 h_{\text{ext}} \, d^3x - \frac{t^2}{2} \int_{\mathbb{T}_\ell^2} |\nabla^{1/2} \bar{m}_3|^2 \, d^2x, \end{aligned} \quad (6.1)$$

where $\bar{m}(x') = \frac{1}{t} \int_0^t m(x', x_3) \, dx_3$ denotes the e_3 -average of the magnetization over $(0, t)$.

Note that for two-dimensional magnetizations (6.1) also holds in the reversed direction if $-C$ is replaced by C . Hence the lower bound is asymptotically sharp. We also remark that a similar sharp estimate for the three-dimensional dipolar energy holds for thin three-dimensional domains in the whole space [53].

For the proof of Lemma 6.1, which is deferred until the end of this section, we need several estimates presented in the following sections.

6.1 Approximation of m by its e_3 -average \bar{m}

Since the thickness t of the film is small, the exchange energy strongly penalizes oscillations of the magnetization in the normal direction of the film. Hence the averaged

magnetization \bar{m} is a good approximation of m , and Assumption (i) in section 2.1 can be made rigorous by the following Poincaré-type inequality

$$\int_{\mathbb{T}_\ell^2 \times (0,t)} |m - \chi_{(0,t)} \bar{m}|^2 d^3x \lesssim t^2 \int_{\mathbb{T}_\ell^2 \times (0,t)} |\partial_3 m|^2 d^3x, \quad (6.2)$$

which holds for all $m \in H^1(\mathbb{T}_\ell^2 \times (0,t); \mathbb{R}^3)$ and can be proved by standard methods.

6.2 Approximation of the stray field energy

In this section, we establish an approximation of the stray field, i.e. a rigorous version of Assumption (i). In particular, we show that for thin films, the difference between the stray field energy of the averaged magnetization and the stray field energy of the full magnetization may be estimated by the exchange energy at lower order. We mention that an estimate of this type already occurred in [41], however, it is not strong enough for our purpose. The statement of Theorem 6.2 below is slightly stronger than what is necessary to prove Lemma 6.1 and might be of independent interest for other thin film regimes.

Theorem 6.2. *Let $m \in H^1(\mathbb{T}_\ell^2 \times (0,t); \mathbb{R}^3)$, then the stray field energy (see (2.2)) satisfies*

$$\left| \int_{\mathbb{T}_\ell^2 \times \mathbb{R}} |h[m]|^2 d^3x - \int_{\mathbb{T}_\ell^2 \times \mathbb{R}} |h[m_3 e_3]|^2 d^3x - \int_{\mathbb{T}_\ell^2 \times \mathbb{R}} |h[m']|^2 d^3x \right| \lesssim t^2 \int_{\mathbb{T}_\ell^2 \times (0,t)} |\nabla m|^2 d^3x, \quad (6.3)$$

$$\left| \int_{\mathbb{T}_\ell^2 \times \mathbb{R}} |h[m]|^2 d^3x - \int_{\mathbb{T}_\ell^2} |h[\chi_{(0,t)} \bar{m}]|^2 d^3x \right| \lesssim t^2 \int_{\mathbb{T}_\ell^2 \times (0,t)} |\nabla m|^2 d^3x, \quad (6.4)$$

where $m' = m - m_3 e_3$ is understood to have values in \mathbb{R}^3 with e_3 -component 0. Moreover, the contributions due to m_3 and m' may be approximated by

$$\left| \int_{\mathbb{T}_\ell^2 \times \mathbb{R}} |h[m_3 e_3]|^2 d^3x - \int_{\mathbb{T}_\ell^2 \times (0,t)} m_3^2 d^3x + \frac{t^2}{2} \int_{\mathbb{T}_\ell^2} |\nabla^{1/2} \bar{m}_3|^2 d^2x \right| \lesssim t^2 \int_{\mathbb{T}_\ell^2 \times (0,t)} |\nabla m|^2 d^3x, \quad (6.5)$$

$$\left| \int_{\mathbb{T}_\ell^2 \times \mathbb{R}} |h[m']|^2 d^3x - \frac{t^2}{2} \int_{\mathbb{T}_\ell^2} |\nabla^{-1/2} \nabla' \cdot \bar{m}'|^2 d^2x \right| \lesssim t^2 \int_{\mathbb{T}_\ell^2 \times (0,t)} |\nabla m|^2 d^3x, \quad (6.6)$$

$$\int_{\mathbb{T}_\ell^2 \times \mathbb{R}} |h[m']|^2 d^3x \lesssim t^2 \int_{\mathbb{T}_\ell^2 \times (0,t)} (|\nabla m|^2 + |m'|^2) d^3x. \quad (6.7)$$

Proof. It is sufficient to argue for $m \in C_c^\infty(\mathbb{T}_\ell^2 \times \mathbb{R}; \mathbb{R}^3)$, because the general case follows by an approximation argument, as we now explain. Since $\mathbb{T}_\ell^2 \times (0, t)$ is an extension domain, there exists, for every $m \in H^1(\mathbb{T}_\ell^2 \times (0, t); \mathbb{R}^3)$, a sequence $(m_n)_{n \in \mathbb{N}}$ with $m_n \in C_c^\infty(\mathbb{T}_\ell^2 \times \mathbb{R}; \mathbb{R}^3)$ such that $\|m - m_n\|_{L^2(\mathbb{T}_\ell^2 \times \mathbb{R})} + \|\nabla m - \nabla m_n\|_{L^2(\mathbb{T}_\ell^2 \times (0, t))} \rightarrow 0$. It remains to check that all terms in (6.3) – (6.6) are continuous. Note that by (6.2), we also have $\|\overline{m}_n - \overline{m}\|_{L^2(\mathbb{T}_\ell^2)} \rightarrow 0$. Moreover, $t \int_{\mathbb{T}_\ell} |\nabla \overline{m}_n|^2 d^2x \lesssim \int_{\mathbb{T}_\ell \times (0, t)} |\nabla m_n|^2 d^3x$ (see (A.11) in the Appendix for a proof). Hence the convergence follows from the elliptic estimate $\int_{\mathbb{T}_\ell^2 \times \mathbb{R}} |h[m_n - m]|^2 d^3x \leq \int_{\mathbb{T}_\ell^2 \times \mathbb{R}} |m_n - m|^2 d^3x$ and by interpolation for the terms involving fractional derivatives.

We write the stray field energy in terms of the magnetostatic potential ϕ

$$\int_{\mathbb{T}_\ell^2 \times \mathbb{R}} |h[m]|^2 d^3x = - \int_{\mathbb{T}_\ell^2 \times \mathbb{R}} \phi \nabla \cdot m d^3x \quad \text{where } \Delta \phi = \nabla \cdot m \text{ in } \mathcal{D}'(\mathbb{T}_\ell^2 \times \mathbb{R}). \quad (6.8)$$

Upon passing to Fourier series (with respect to the in-plane variables), we get

$$\int_{\mathbb{T}_\ell^2 \times \mathbb{R}} \phi \nabla \cdot m d^3x = \frac{1}{\ell^2} \int_{\mathbb{R}} \sum_{k \in \frac{2\pi}{\ell} \mathbb{Z}^2} \widehat{\phi}_k^*(z) (\partial_z \widehat{m}_{3,k}(z) - ik \cdot \widehat{m}'_k(z)) dz, \quad (6.9)$$

where the Fourier coefficients $\widehat{\phi}_k : \mathbb{R} \rightarrow \mathbb{C}$ for $k \in \frac{2\pi}{\ell} \mathbb{Z}^2$ of ϕ solve

$$\partial_z^2 \widehat{\phi}_k - |k|^2 \widehat{\phi}_k = \partial_z \widehat{m}_{3,k} - ik \cdot \widehat{m}'_k. \quad (6.10)$$

We introduce the fundamental solution

$$H_k(z) = \begin{cases} \frac{e^{-|k||z|}}{|k|} & \text{for } k \neq 0, \\ -|z| & \text{for } k = 0, \end{cases} \quad (6.11)$$

which satisfies

$$-\partial_z^2 H_k + |k|^2 H_k = 2\delta \quad \text{in } \mathcal{D}'(\mathbb{R}) \text{ for all } k \in \mathbb{Z}^2, \quad (6.12)$$

where δ denotes the Dirac measure at 0. The fundamental solution allows to rewrite $\widehat{\phi}_k(z)$ as

$$\widehat{\phi}_k(z) = -\frac{1}{2} \int_{\mathbb{R}} H_k(z - z') (\partial_z \widehat{m}_{3,k}(z') - ik \cdot \widehat{m}'_k(k, z')) dz', \quad (6.13)$$

which by (6.9) leads to the following expression for the stray field energy

$$\begin{aligned} \int_{\mathbb{T}_\ell^2 \times \mathbb{R}} |h[m]|^2 d^3x &= \frac{1}{2\ell^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \sum_{k \in \frac{2\pi}{\ell} \mathbb{Z}^2} (\partial_z \widehat{m}_{3,k}(z) - ik \cdot \widehat{m}'_k(z))^* \\ &\quad \times H_k(z - z') (\partial_z \widehat{m}_{3,k}(z') - ik \cdot \widehat{m}'_k(z')) dz dz'. \end{aligned} \quad (6.14)$$

To prove (6.3), we need to show that the mixed terms in (6.14), i.e. terms of the form

$$I := \frac{1}{\ell^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \sum_{k \in \frac{2\pi}{\ell} \mathbb{Z}^2} \partial_z \widehat{m}_{3,k}^*(z) H_k(z - z') (ik \cdot \widehat{m}'_k(z')) dz dz' \quad (6.15)$$

satisfy $|I| \lesssim t^2 \int_{\mathbb{T}_{\ell}^2 \times (0,t)} |\nabla m|^2 dx$. Integrating by parts in (6.15), we get

$$I = -\frac{1}{\ell^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \sum_{k \in \frac{2\pi}{\ell} \mathbb{Z}^2} \widehat{m}_{3,k}^*(z) \partial_z H_k(z - z') (ik \cdot \widehat{m}'_k(z')) dz dz'. \quad (6.16)$$

We write $m = \chi_{(0,t)} \overline{m} + u$ where as usual $\overline{m}(x') = \frac{1}{t} \int_0^t m(x', x_3) dx_3$ denotes the average of m over in the e_3 -direction. With this notation, (6.16) turns into

$$I = -\frac{1}{\ell^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \sum_{k \in \frac{2\pi}{\ell} \mathbb{Z}^2} (\chi_{(0,t)}(z) \widehat{\overline{m}}_{3,k} + \widehat{u}_{3,k}(z))^* \partial_z H(k, z - z') \\ \times \left(ik \cdot \chi_{(0,t)}(z') \widehat{\overline{m}}'_k + ik \cdot \widehat{u}'_k(z') \right) dz dz'. \quad (6.17)$$

Since $\partial_z H_k(z) = -\frac{z}{|z|} e^{-|k||z|}$ is anti-symmetric in z , we have $\int_0^t \int_0^t \partial_z H_k(z - z') dz dz' = 0$ which means that upon expanding (6.17), the term involving $\widehat{\overline{m}}_3$ and $\widehat{\overline{m}}'$ vanishes. Furthermore, we have $|\partial_z H_k| \leq 1$ and hence the remaining terms in (6.17) may be estimated by

$$|I| \leq \frac{1}{\ell^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \sum_{k \in \frac{2\pi}{\ell} \mathbb{Z}^2} \left(|\widehat{u}_{3,k}(z)| |k \cdot \widehat{m}'_k(z')| + |\chi_{(0,t)}(z) \widehat{\overline{m}}_{3,k}| |k \cdot \widehat{u}'_k(z')| \right) dz dz'. \quad (6.18)$$

Note that passing to Fourier series in the in-plane variables commutes with taking e_3 -averages. Thus $\widehat{u}_{j,k}$ has e_3 -average zero for all $j = 1, 2, 3$ and the intermediate value theorem yields $\tau_{j,k}, \rho_{j,k} \in (0, t)$ such that $\Re \widehat{u}_{j,k}(\tau_{j,k}) = 0$ and $\Im \widehat{u}_{j,k}(\rho_{j,k}) = 0$. By the fundamental theorem of calculus, we hence get the estimate

$$|\widehat{u}_{j,k}(z)| \lesssim \int_0^t |\partial_z \widehat{m}_{j,k}(\tau)| d\tau \quad \text{for all } z \in (0, t) \text{ and } j = 1, 2, 3. \quad (6.19)$$

Inserting (6.19) into (6.18) and using Jensen's inequality yields the rough estimate

$$|I| \lesssim \sum_{n,j=1}^3 \frac{t}{\ell^2} \int_0^t \int_0^t \sum_{k \in \frac{2\pi}{\ell} \mathbb{Z}^2} |\partial_z \widehat{m}_{j,k}(z)| |k| |\widehat{m}_{n,k}(z')| dz dz'. \quad (6.20)$$

By Young's inequality and Parseval's identity, we conclude

$$|I| \lesssim \sum_{n,j=1}^3 \frac{t}{\ell^2} \int_0^t \int_0^t \sum_{k \in \frac{2\pi}{\ell} \mathbb{Z}^2} (|\partial_z \widehat{m}_{j,k}(z)|^2 + |k|^2 |\widehat{m}_{n,k}(z')|^2) dz dz' \\ \lesssim t^2 \int_{\mathbb{T}_{\ell}^2 \times (0,t)} |\nabla m|^2 dx, \quad (6.21)$$

which completes the proof of (6.3). Assuming for a moment that (6.5) and (6.6) hold, identity (6.4) is obtained as follows. Applying (6.3) to m and $\chi_{(0,t)}\bar{m}$, we get

$$\begin{aligned} & \left| \int_{\mathbb{T}_\ell^2} |h[m]|^2 d^3x - \int_{\mathbb{T}_\ell^2} |h[\chi_{(0,t)}\bar{m}]|^2 d^3x \right. \\ & - \int_{\mathbb{T}_\ell^2 \times \mathbb{R}} |h[m_3 e_3]|^2 d^3x + \int_{\mathbb{T}_\ell^2 \times \mathbb{R}} |h[\chi_{(0,t)}\bar{m}_3 e_3]|^2 d^3x \\ & \left. - \int_{\mathbb{T}_\ell^2 \times \mathbb{R}} |h[m']|^2 d^3x + \int_{\mathbb{T}_\ell^2 \times \mathbb{R}} |h[\chi_{(0,t)}\bar{m}']|^2 d^3x \right| \stackrel{(6.3)}{\lesssim} t^2 \int_{\mathbb{T}_\ell^2 \times (0,t)} |\nabla m|^2 d^3x, \end{aligned} \quad (6.22)$$

where we have also used (see (A.11) in the appendix for a proof)

$$\int_{\mathbb{T}_\ell^2 \times (0,t)} |\nabla (\chi_{(0,t)}\bar{m})|^2 d^3x = t \int_{\mathbb{T}_\ell^2} |\nabla' \bar{m}|^2 d^2x \stackrel{(A.11)}{\leq} \int_{\mathbb{T}_\ell^2 \times (0,t)} |\nabla m|^2 d^3x.$$

Applying (6.5) and (6.6) to (6.22) yields the claim

$$\begin{aligned} & \left| \int_{\mathbb{T}_\ell^2} |h[m]|^2 d^3x - \int_{\mathbb{T}_\ell^2} |h[\chi_{(0,t)}\bar{m}]|^2 d^3x \right. \\ & \left. - \int_{\mathbb{T}_\ell^2 \times (0,t)} m_3^2 d^3x + \int_{\mathbb{T}_\ell^2 \times (0,t)} (\chi_{(0,t)}\bar{m}_3)^2 d^2x \right| \stackrel{(6.5),(6.6)}{\lesssim} t^2 \int_{\mathbb{T}_\ell^2 \times (0,t)} |\nabla m|^2 d^3x \quad (6.23) \\ & \left| \int_{\mathbb{T}_\ell^2} |h[m]|^2 d^3x - \int_{\mathbb{T}_\ell^2} |h[\chi_{(0,t)}\bar{m}]|^2 d^3x \right| \stackrel{(6.2)}{\lesssim} t^2 \int_{\mathbb{T}_\ell^2 \times (0,t)} |\nabla m|^2 d^3x. \end{aligned}$$

We turn to the proof of (6.5). Integrating by parts twice and inserting (6.12), we get

$$\begin{aligned} & \int_{\mathbb{T}_\ell^2 \times \mathbb{R}} |h[m_3]|^2 d^3x \stackrel{(6.14)}{=} \frac{1}{2\ell^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \sum_{k \in \frac{2\pi}{\ell} \mathbb{Z}^2} \partial_z \hat{m}_{3,k}^*(z) H_k(z-z') \partial_z \hat{m}_{3,k}(z') dz dz' \\ & = -\frac{1}{2\ell^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \sum_{k \in \frac{2\pi}{\ell} \mathbb{Z}^2} \hat{m}_{3,k}^*(z) \partial_z^2 H_k(z-z') \hat{m}_{3,k}(z') dz dz' \\ & \stackrel{(6.12)}{=} \frac{1}{\ell^2} \int_{\mathbb{R}} \sum_{k \in \frac{2\pi}{\ell} \mathbb{Z}^2} |\hat{m}_{3,k}(z)|^2 dz \\ & \quad - \frac{1}{2\ell^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \sum_{k \in \frac{2\pi}{\ell} \mathbb{Z}^2} \hat{m}_{3,k}^*(z) |k| e^{-|k||z-z'|} \hat{m}_{3,k}(z') dz dz' \end{aligned}$$

Since $|1 - e^{-|k||z|}| \leq |k|t$ for $z \in (-t, t)$, the last line above

$$J := \frac{1}{2\ell^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \sum_{k \in \frac{2\pi}{\ell} \mathbb{Z}^2} \hat{m}_{3,k}^*(z) |k| e^{-|k||z-z'|} \hat{m}_{3,k}(z') dz dz' \quad (6.24)$$

may be estimated, with the help of Young's inequality, by

$$\begin{aligned} \left| J - \frac{t^2}{2\ell^2} \sum_{k \in \frac{2\pi}{\ell}\mathbb{Z}^2} |k| |\widehat{m}_{3,k}(z)|^2 \right| &\lesssim \frac{t}{\ell^2} \int_0^t \int_0^t \sum_{k \in \frac{2\pi}{\ell}\mathbb{Z}^2} |\widehat{m}_{3,k}(z)| |k|^2 |\widehat{m}_{3,k}(z')| \, dz \, dz' \\ &\lesssim \frac{t^2}{\ell^2} \int_0^t \sum_{k \in \frac{2\pi}{\ell}\mathbb{Z}^2} |k|^2 |\widehat{m}_{3,k}(z)|^2 \, dz, \end{aligned}$$

which by Parseval's identity is equivalent to

$$\left| J - \frac{t^2}{2} \int_{\mathbb{T}_\ell^2} |\nabla^{1/2} \overline{m}_3|^2 \, d^2x \right| \lesssim t^2 \int_{\mathbb{T}_\ell^2 \times (0,t)} |\nabla' m_3|^2 \, d^3x. \quad (6.25)$$

In total, we get

$$\begin{aligned} \left| \int_{\mathbb{T}_\ell^2 \times \mathbb{R}} |h[m_3 e_3]|^2 \, d^3x - \int_{\mathbb{T}_\ell^2 \times (0,t)} m_3^2 \, d^3x + \frac{t^2}{2} \int_{\mathbb{T}_\ell^2} |\nabla^{1/2} \overline{m}_3|^2 \, d^3x \right| \\ \lesssim t^2 \int_{\mathbb{T}_\ell^2 \times (0,t)} |\nabla' m_3|^2 \, d^3x, \end{aligned} \quad (6.26)$$

which proves (6.5). We continue with the proof of (6.6). Since $|1 - e^{-|k||z|}| \leq |k|t$ for $z \in (0, t)$, we may insert $|H_k(z - z') - \frac{1}{|k|}| \leq t$ for $k \neq 0$ into (6.14)

$$\int_{\mathbb{T}_\ell^2 \times \mathbb{R}} |h[m']|^2 \, d^3x \stackrel{(6.14)}{=} \frac{1}{2\ell^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \sum_{k \in \frac{2\pi}{\ell}\mathbb{Z}^2 \setminus \{0\}} (k \cdot \widehat{m}'_k(z))^* H_k(z - z') k \cdot \widehat{m}'_k(z') \, dz \, dz'. \quad (6.27)$$

This yields

$$\left| \int_{\mathbb{T}_\ell^2 \times \mathbb{R}} |h[m']|^2 \, d^3x - \frac{t}{2\ell^2} \sum_{k \in \frac{2\pi}{\ell}\mathbb{Z}^2 \setminus \{0\}} \frac{|k \cdot \widehat{m}'_k|^2}{|k|} \right| \lesssim \frac{t^2}{2\ell^2} \int_{\mathbb{R}} \sum_{k \in \frac{2\pi}{\ell}\mathbb{Z}^2} |k \cdot \widehat{m}'_k(z)|^2 \, dz, \quad (6.28)$$

which proves the first equality. The second equality follows as in (2.4). \square

Proof of Lemma 6.1. We invoke Theorem 6.2 to obtain a lower bound for the stray field energy. Combining (6.3) with (6.5) and neglecting the non-negative term $\int_{\mathbb{T}_\ell^2 \times \mathbb{R}} |h[m']|^2 \, d^3x$, we get

$$\begin{aligned} \int_{\mathbb{T}_\ell^2} |h[m]|^2 \, d^3x &\stackrel{(6.3)}{\geq} \int_{\mathbb{T}_\ell^2 \times \mathbb{R}} |h[m_3 e_3]|^2 \, d^3x - Ct^2 \int_{\mathbb{T}_\ell^2 \times (0,t)} |\nabla m|^2 \, d^3x \\ &\stackrel{(6.5)}{\geq} \int_{\mathbb{T}_\ell^2 \times (0,t)} m_3^2 \, d^3x - \frac{t^2}{2} \int_{\mathbb{T}_\ell^2} |\nabla^{1/2} \overline{m}_3|^2 \, d^2x \\ &\quad - Ct^2 \int_{\mathbb{T}_\ell^2 \times (0,t)} |\nabla m|^2 \, d^3x, \end{aligned} \quad (6.29)$$

for some universal constant $C > 0$. Note that estimating $\int_{\mathbb{T}_\ell^2 \times \mathbb{R}} |h[m']|^2 d^3x$ by zero is reasonable, since (6.6) shows that the term is controlled by the exchange and anisotropy energy at lower order. Inserting (6.29) into the energy E yields

$$\begin{aligned}
E[m] &\stackrel{(2.1)}{=} \int_{\mathbb{T}_\ell^2 \times (0,t)} (|\nabla m|^2 + Q(m_1^2 + m_2^2) - 2m_3 h_{\text{ext}}) d^3x + \int_{\mathbb{T}_\ell^2 \times \mathbb{R}} |h|^2 d^3x. \\
&\stackrel{(6.29)}{\geq} \int_{\mathbb{T}_\ell^2 \times (0,t)} (|\nabla m|^2 + Q(m_1^2 + m_2^2) - 2m_3 h_{\text{ext}}) d^3x + \int_{\mathbb{T}_\ell^2 \times (0,t)} m_3^2 d^2x \\
&\quad - \frac{t^2}{2} \int_{\mathbb{T}_\ell^2} |\nabla^{1/2} \bar{m}_3|^2 d^2x - Ct^2 \int_{\mathbb{T}_\ell^2 \times (0,t)} |\nabla m|^2 d^3x.
\end{aligned} \tag{6.30}$$

The constraint $|m| = 1$ allows to combine the leading order of the stray field energy with the anisotropy energy which leads to constant contribution and a renormalized anisotropy term

$$\int_{\mathbb{T}_\ell^2 \times (0,t)} Q(m_1^2 + m_2^2) dx + t \int_{\mathbb{T}_\ell^2} m_3^2 dx = \ell^2 t + \int_{\mathbb{T}_\ell^2 \times (0,t)} (Q-1)(m_1^2 + m_2^2) dx. \tag{6.31}$$

Finally, we insert (6.31) into (6.30) to extract the leading order constant $\ell^2 t$ and conclude the claim of Lemma 6.1

$$\begin{aligned}
E[m] &\geq \ell^2 t + \int_{\mathbb{T}_\ell^2 \times (0,t)} (|\nabla m|^2 + (Q-1)(m_1^2 + m_2^2) - 2m_3 h_{\text{ext}}) d^3x \\
&\quad - \frac{t^2}{2} \int_{\mathbb{T}_\ell^2} |\nabla^{1/2} \bar{m}_3|^2 d^2x - Ct^2 \int_{\mathbb{T}_\ell^2 \times (0,t)} |\nabla m|^2 d^3x,
\end{aligned} \tag{6.32}$$

which completes the proof. \square

7 Proofs for the full energy E

The proofs for the full energy E are based on the arguments in the proofs for the reduced energy F . We recommend to read section 5 first.

Under mild assumptions on ℓ, t, Q and h_{ext} , weaker than those of Theorems 3.1 – 3.4, Lemma 4.1 and Theorem 6.2 yield the following estimates for the rescaled energy J .

Lemma 7.1. *There are universal constants $C, \delta > 0$ such that for $(\ell, t, Q, h_{\text{ext}})$ which satisfy*

$$Q > 1, \quad t < \delta \min\{1, \ell\} \quad \text{and} \quad \frac{\ell}{\sqrt{Q-1}} h_{\text{ext}}(\ell x') = g(x') \tag{7.1}$$

for some $g \in L^1(\mathbb{T}^2)$, the rescaled energy J (see (3.3)) satisfies

$$\begin{aligned}
J[m] &\geq \left(1 - Ct^2 - \frac{t}{4\sqrt{Q-1}}\right) \int_{\mathbb{T}^2 \times (0,1)} \left(\varepsilon |\nabla m|^2 + \frac{1}{\varepsilon} (m_1^2 + m_2^2) \right) d^3x \\
&\quad + \frac{1}{2\varepsilon t^2(Q-1)} \int_{\mathbb{T}^2 \times (0,1)} |\partial_3 m|^2 d^3x - 2 \int_{\mathbb{T}^2} g \bar{m}_3 d^2x \\
&\quad - \frac{t}{\pi\sqrt{Q-1}} \log \left(c_* \max \left\{ 1, \min \left\{ \frac{1}{\varepsilon \int_{\mathbb{T}^2} |\nabla m_3| dx}, \frac{1}{\varepsilon} \right\} \right\} \right) \int_{\mathbb{T}^2} |\nabla \bar{m}_3| d^2x,
\end{aligned} \tag{7.2}$$

for all $m \in H^1(\mathbb{T}^2 \times (0,1); \mathbb{S}^2)$, where we have abbreviated $\varepsilon := \frac{1}{\ell\sqrt{Q-1}}$. Furthermore, for any $\bar{m} \in H^1(\mathbb{T}^2; \mathbb{S}^2)$ we have the upper bound

$$\begin{aligned}
J[\chi_{(0,1)} \bar{m}] &\leq (1 + Ct^2) \int_{\mathbb{T}^2} \left(\varepsilon |\nabla \bar{m}|^2 + \frac{1}{\varepsilon} (\bar{m}_1^2 + \bar{m}_2^2) \right) d^3x \\
&\quad - 2 \int_{\mathbb{T}^2} \bar{m}_3 g d^2x - \frac{t}{2\sqrt{Q-1}} \int_{\mathbb{T}^2} |\nabla^{1/2} \bar{m}_3|^2 d^2x.
\end{aligned} \tag{7.3}$$

Proof. The lower bound for E in Lemma 6.1 implies a lower bound for the rescaled energy J

$$\begin{aligned}
J[m] &= \frac{E[m(\ell \cdot, \ell \cdot, t \cdot)] - \ell^2 t}{\ell t \sqrt{Q-1}} \stackrel{(6.1)}{\geq} (1 - Ct^2) \int_{\mathbb{T}^2 \times (0,1)} \left(\frac{1}{\ell\sqrt{Q-1}} |\nabla' m|^2 \right. \\
&\quad \left. + \frac{\ell}{t^2\sqrt{Q-1}} |\partial_3 m|^2 + \ell\sqrt{Q-1} (m_1^2 + m_2^2) \right) d^3x \\
&\quad - \frac{2\ell}{\sqrt{Q-1}} \int_{\mathbb{T}^2} \bar{m}_3(x') h_{\text{ext}}(\ell x') d^2x - \frac{t}{2\sqrt{Q-1}} \int_{\mathbb{T}^2} |\nabla^{1/2} \bar{m}_3|^2 d^2x.
\end{aligned} \tag{7.4}$$

We insert

$$\frac{\ell}{\sqrt{Q-1}} h_{\text{ext}}(\ell x') = g(x') \quad \text{and} \quad \varepsilon = \frac{1}{\ell\sqrt{Q-1}}$$

to obtain

$$\begin{aligned}
J[m] &\stackrel{(6.1)}{\geq} (1 - Ct^2) \int_{\mathbb{T}^2 \times (0,1)} \left(\varepsilon |\nabla' m|^2 + \frac{1}{\varepsilon t^2(Q-1)} |\partial_3 m|^2 + \frac{1}{\varepsilon} (m_1^2 + m_2^2) \right) d^3x \\
&\quad - 2 \int_{\mathbb{T}^2} g \bar{m}_3 d^2x - \frac{t}{2\sqrt{Q-1}} \int_{\mathbb{T}^2} |\nabla^{1/2} \bar{m}_3|^2 d^2x.
\end{aligned} \tag{7.5}$$

In view of (7.1) we may assume that

$$(1 - Ct^2) \left(\frac{1}{\varepsilon t^2(Q-1)} - \varepsilon \right) \geq \frac{1}{2\varepsilon t^2(Q-1)}. \quad (7.6)$$

Hence, applying Lemma 4.1 to the last term in (7.5) and inserting (7.6) we arrive at (7.2). The proof for the upper bound (7.3) is simpler and analogous to the arguments that led to (7.5). \square

7.1 Proof of Theorem 3.1

It is possible to invoke the lower bound for F on slices $\{x_3 = \text{const}\}$ to obtain the lower bound for the full (rescaled) energy J . However, we will not pursue this option. Instead, we apply the $H^{1/2}$ -bound of Lemma 4.1 directly and extend the arguments of the previous section. The reason is related to the fact that $C^\infty(\mathbb{T}^2 \times (0, 1); \mathbb{S}^2)$ is not dense in $H^1(\mathbb{T}^2 \times (0, 1); \mathbb{S}^2)$, which can be seen by considering $f(x) = \frac{x}{|x|}$ (see [6, 5, 28]). Hence, evaluating Sobolev functions on slices $\{x_3 = \text{const}\}$ and confirming that the constraint $|m| = 1$ still holds requires to use the precise representative of a Sobolev function and gets rather technical.

Proof of the lower bound and compactness in Theorem 3.1. Our starting point is the lower bound (7.2). It turns out to be more convenient to use the parameter $\varepsilon = \frac{1}{\ell\sqrt{Q-1}}$ instead of ℓ . We first note that for $\varepsilon < 1$ the last term in (7.2) may be estimated with the aid of (A.11) and (A.10) by

$$\begin{aligned} & \log \left(c_* \max \left\{ 1, \min \left\{ \frac{1}{\varepsilon \int_{\mathbb{T}^2} |\nabla m_3| \, dx}, \frac{1}{\varepsilon} \right\} \right\} \right) \int_{\mathbb{T}^2} |\nabla \bar{m}_3| \, d^2x \\ & \leq \log(c_*/\varepsilon) \int_{\mathbb{T}^2 \times (0,1)} \left(\varepsilon |\nabla m|^2 + \frac{1}{\varepsilon} (m_1^2 + m_2^2) \right) d^3x. \end{aligned} \quad (7.7)$$

For Q and $(\ell_k, t_k, h_{\text{ext},k})$ satisfying (3.4), we abbreviate

$$\varepsilon_k := \frac{1}{\ell_k \sqrt{Q-1}} \rightarrow 0 \quad \text{and} \quad g_k := \frac{\ell_k}{\sqrt{Q-1}} h_{\text{ext},k}(\ell_k \cdot) \rightarrow g, \quad (7.8)$$

and note that

$$t_k^2 + \frac{t_k}{\sqrt{Q-1}} \xrightarrow{(3.4)} 0. \quad (7.9)$$

Inserting (3.4) and (7.7) – (7.9) into the lower bound (7.2), we deduce that for any $\gamma > 0$

and sufficiently large $k \geq k_0(\gamma)$, we have

$$\begin{aligned} J_k[m] &\geq \left(1 - \frac{\lambda}{\lambda_c} - \gamma\right) \int_{\mathbb{T}^2 \times (0,1)} \left(\varepsilon_k |\nabla m|^2 + \frac{1}{\varepsilon_k} (m_1^2 + m_2^2)\right) d^3x \\ &\quad + \frac{1}{2\varepsilon_k t_k^2 (Q-1)} \int_{\mathbb{T}^2 \times (0,1)} |\partial_3 m|^2 d^3x - 2 \int_{\mathbb{T}^2} \bar{m}_3 g_k d^2x. \end{aligned} \quad (7.10)$$

Note that (7.10) for $2\gamma \leq 1 - \frac{\lambda}{\lambda_c}$ and sufficiently large k implies

$$\int_{\mathbb{T}^2 \times (0,1)} (m_1^2 + m_2^2) d^3x \lesssim \frac{\varepsilon_k}{(\lambda_c - \lambda)} (J_k[m] + 2\|g_k\|_{L^1}). \quad (7.11)$$

Using Poincaré's inequality and (7.10) for $\gamma < 1 - \frac{\lambda}{\lambda_c}$ again, we get

$$\begin{aligned} \int_{\mathbb{T}^2 \times (0,1)} |m - \chi_{(0,1)} \bar{m}|^2 d^3x &\lesssim \int_{\mathbb{T}^2 \times (0,1)} |\partial_3 m|^2 d^3x \\ &\stackrel{(7.10)}{\lesssim} \varepsilon_k t_k^2 (Q-1) \left(\limsup_{k \rightarrow \infty} J_k[m] + 2\|g_k\|_{L^1} \right). \end{aligned} \quad (7.12)$$

Furthermore, applying (A.10) and (A.11) to (7.10) again implies the lower bound

$$J[m] \geq 2 \left(1 - \frac{\lambda}{\lambda_c} - \gamma\right) \int_{\mathbb{T}^2} |\nabla' \bar{m}_3| d^2x - 2 \int_{\mathbb{T}^2} \bar{m}_3 g_k d^2x. \quad (7.13)$$

In order to prove compactness, let $m^{(k)} \in H^1(\mathbb{T}^2 \times (0,1); \mathbb{S}^2)$ with $\limsup_{k \rightarrow \infty} J[m_k] < \infty$. Since $\lambda < \lambda_c$ and $g_k \rightarrow g$ in $L^1(\mathbb{T}^2)$, inequality (7.13) implies a uniform bound on $\bar{m}_3^{(k)}$ in $BV(\mathbb{T}^2)$. A standard compactness argument implies that $\bar{m}_3^{(k)} \rightarrow \bar{m}_3$ in $L^1(\mathbb{T}^2)$ for a subsequence (not relabelled) and some $\bar{m}_3 \in BV(\mathbb{T}^2)$. We will now show that in fact $m^{(k)} \rightarrow \chi_{(0,1)} \bar{m}_3 e_3$ in $L^1(\mathbb{T}^2 \times (0,1); \mathbb{R}^3)$. Indeed, the triangle inequality yields

$$\begin{aligned} \int_{\mathbb{T}^2 \times (0,1)} |m^{(k)} - \chi_{(0,1)} \bar{m}_3 e_3| d^3x &\leq \int_{\mathbb{T}^2 \times (0,1)} \left(|m_1^{(k)}|^2 + |m_2^{(k)}|^2\right)^{1/2} d^3x \\ &\quad + \int_{\mathbb{T}^2 \times (0,1)} |m_3^{(k)} - \chi_{(0,1)} \bar{m}_3^{(k)}| d^3x + \int_{\mathbb{T}^2} |\bar{m}_3^{(k)} - \bar{m}_3| d^2x, \end{aligned} \quad (7.14)$$

and we already know that the last term on the right hand side of (7.14) vanishes. Furthermore, the first term vanishes due to (7.11) and the second one due to (7.12) and (3.4). This completes the proof of the compactness statement.

The liminf inequality is easily obtained from the lower bound (7.13). Indeed, let $m^{(k)} \in H^1(\mathbb{T}^2 \times (0,1); \mathbb{S}^2)$ with $m^{(k)} \rightarrow m$ in $L^1(\mathbb{T}^2 \times (0,1))$. By Jensen's inequality, we also

have $\overline{m}^{(k)} \rightarrow \overline{m}$ in $L^1(\mathbb{T}^2)$. By lower semicontinuity of the BV seminorm and since γ was arbitrary, we obtain from (7.13) in the limit

$$\liminf_{k \rightarrow \infty} J_k[m^{(k)}] \geq \left(1 - \frac{\lambda}{\lambda_c}\right) \int_{\mathbb{T}^2} |\nabla' \overline{m}_3| \, d^2x - 2 \int_{\mathbb{T}^2} \overline{m}_3 g \, d^2x.$$

□

It remains to prove the upper bound for the Γ -convergence. As it turns out, we may use the recovery sequence for the reduced energy F also for the full energy E (up to thickening).

Construction of the recovery sequence in Theorem 3.1. Let $\lambda \leq \lambda_c$ and $\overline{m} \in BV(\mathbb{T}^2; \{\pm e_3\})$. Furthermore, let $\overline{m}_\varepsilon \in H^1(\mathbb{T}^2; \mathbb{S}^2)$ denote the recovery sequence for $F_{\varepsilon, \lambda}$ from Lemma 5.3. With the notation (7.8) we set

$$m^{(k)}(x', x_3) := \chi_{(0,1)}(x_3) \overline{m}_{\varepsilon_k}(x') \quad \text{for } (x', x_3) \in \mathbb{T}^2 \times (0, 1) \quad (7.15)$$

and claim that

$$\limsup_{k \rightarrow \infty} J_k[m^{(k)}] \leq J_*[\overline{m}]. \quad (7.16)$$

Inserting the abbreviation $\lambda_k := \frac{t_k |\log(\varepsilon_k)|}{4\sqrt{Q-1}}$ into the upper bound (7.3), we obtain

$$\begin{aligned} J_k[m^{(k)}] &\leq (1 + Ct_k^2) \int_{\mathbb{T}^2} \left(\varepsilon_k |\nabla \overline{m}_{\varepsilon_k}|^2 + \frac{1}{\varepsilon_k} (\overline{m}_{\varepsilon_k,1}^2 + \overline{m}_{\varepsilon_k,2}^2) \right) d^2x \\ &\quad - \frac{2\lambda_k}{|\log \varepsilon_k|} \int_{\mathbb{T}^2} |\nabla^{1/2} \overline{m}_{\varepsilon_k,3}|^2 - 2 \int_{\mathbb{T}^2} g_k \overline{m}_{\varepsilon_k,3} \, d^2x \\ &= 2F_{\varepsilon_k, \lambda_k}[\overline{m}_{\varepsilon_k}] - 2 \int_{\mathbb{T}^2} g_k \overline{m}_{\varepsilon_k,3} \, d^2x + Ct_k^2 \int_{\mathbb{T}^2} \left(\varepsilon_k |\nabla \overline{m}_{\varepsilon_k}|^2 + \frac{1}{\varepsilon_k} (\overline{m}_{\varepsilon_k,1}^2 + \overline{m}_{\varepsilon_k,2}^2) \right) d^2x. \end{aligned} \quad (7.17)$$

We have shown in Lemma 5.3 that

$$\int_{\mathbb{T}^2} \left(\varepsilon_k |\nabla \overline{m}_{\varepsilon_k}|^2 + \frac{1}{\varepsilon_k} (\overline{m}_{\varepsilon_k,1}^2 + \overline{m}_{\varepsilon_k,2}^2) \right) d^2x \rightarrow 2 \int_{\mathbb{T}^2} |\nabla \overline{m}_3| \, d^2x < \infty. \quad (7.18)$$

Since (3.4) implies $t_k \rightarrow 0$, $\lambda_k \rightarrow \lambda < \lambda_c$ and $g_k \rightarrow g$ in $L^1(\mathbb{T}^2)$, the claim follows upon applying Lemma 5.3 to (7.17)

$$\limsup_{k \rightarrow \infty} J_k[m^{(k)}] \leq 2F_{*, \lambda}[\overline{m}] - 2 \int_{\mathbb{T}^2} g \overline{m}_3 \, d^2x. \quad (7.19)$$

□

7.2 Proof of Theorem 3.2

Proof of Theorem 3.2. We begin with the proof of the lower bound for which we use (7.2) with $g = 0$. For sufficiently small δ , the regime (3.10) implies

$$Ct^2 + \frac{t}{\sqrt{Q-1}} \stackrel{(3.10)}{\lesssim} C\delta^2 + \delta \lesssim \delta. \quad (7.20)$$

Analogous to the argument that lead from (5.38) to (5.40), but now with (7.20) instead of (5.33), we reduce (7.2) to the case

$$\begin{aligned} J[m] &\geq \left(1 - Ct^2 - \frac{t}{\sqrt{Q-1}}\right) \int_{\mathbb{T}^2 \times (0,1)} \left(\varepsilon |\nabla m|^2 + \frac{1}{\varepsilon}(m_1^2 + m_2^2)\right) d^3x \\ &\quad + \frac{1}{2\varepsilon t^2(Q-1)} |\partial_3 m|^2 d^3x - \frac{t \log\left(c_* \frac{1}{\varepsilon \int_{\mathbb{T}^2} |\nabla \bar{m}_3| dx}\right)}{\pi \sqrt{Q-1}} \int_{\mathbb{T}^2} |\nabla \bar{m}_3| d^2x. \end{aligned} \quad (7.21)$$

Abbreviating the energetic cost for m to deviate from the optimal Bloch wall profile by

$$D_\varepsilon[m] := \int_{\mathbb{T}^2 \times (0,1)} \left(\varepsilon |\nabla m|^2 + \frac{1}{\varepsilon}(1 - m_3^2)\right) d^3x - 2 \int_{\mathbb{T}^2} |\nabla \bar{m}_3| d^2x, \quad (7.22)$$

and inserting $\mu := \varepsilon e^{2\pi t^{-1}\sqrt{Q-1}} \int_{\mathbb{T}^2} |\nabla \bar{m}_3| dx$ and $c_{**} := c_* e^{2\pi(1+Ct\sqrt{Q-1})} \stackrel{(3.10)}{\lesssim} 1$ into the lower bound (7.21) we get

$$\begin{aligned} J[m] &\geq \left(1 - Ct^2 - \frac{t}{\sqrt{Q-1}}\right) D_\varepsilon[m] + \frac{1}{2\varepsilon t^2(Q-1)} \int_{\mathbb{T}^2 \times (0,1)} |\partial_3 m|^2 d^3x \\ &\quad - \frac{\log(c_{**}/\mu)}{\pi} \mu t \ell e^{-2\pi t^{-1}\sqrt{Q-1}}. \end{aligned} \quad (7.23)$$

Minimizing in $\mu > 0$ then yields the lower bound

$$J[m] \gtrsim -c_{**} t \ell e^{-2\pi t^{-1}\sqrt{Q-1}} \gtrsim -t \ell e^{-2\pi t^{-1}\sqrt{Q-1}}. \quad (7.24)$$

It remains to construct a sequence that achieves the optimal scaling. Let $m_{\varepsilon,N}$ denote the function constructed in Lemma 5.5 and define $m_{\varepsilon,N} := \chi_{(0,1)} \bar{m}_{\varepsilon,N}$. We insert (5.49) and (5.50) into (7.3) and use that (3.10) implies $t^2 \lesssim \frac{t}{\sqrt{Q-1}}$ to deduce

$$J[m_{\varepsilon,N}] \leq 4N \left(1 + Ct^2 - \frac{t \log\left(\frac{c}{2\varepsilon N}\right)}{2\pi \sqrt{Q-1}}\right) \stackrel{(3.10)}{\leq} 4N \left(1 - \frac{t \log\left(\frac{\tilde{c}}{2\varepsilon N}\right)}{2\pi \sqrt{Q-1}}\right) \quad (7.25)$$

for some universal $\tilde{c} > 0$. Optimizing in N leads to

$$N := 2 \left\lceil \ell \sqrt{Q-1} \frac{e^{-2\pi t^{-1}\sqrt{Q-1}}}{K} \right\rceil, \quad (7.26)$$

which satisfies $N \geq 2$ due to (3.10) and is hence admissible. Inserting (7.26) into (7.25), and taking $K \geq \frac{8}{\varepsilon}$, we conclude that the function $m_{\varepsilon, N}$ indeed achieves the optimal scaling

$$J[m_{\varepsilon, N}] \lesssim -t\ell e^{-2\pi t^{-1}\sqrt{Q-1}}.$$

□

7.3 Proof of Proposition 3.3

Proof of Proposition 3.3. Let m satisfy (3.12). Then (7.20) and (7.23) imply $\mu \sim 1$ and hence (3.15)

$$\int_{\mathbb{T}^2} |\nabla \bar{m}_3| \, d^2x \sim \ell \sqrt{Q-1} e^{-2\pi t^{-1}\sqrt{Q-1}},$$

where here and throughout the rest of this proof, the constants associated with \lesssim, \gtrsim and \sim may depend on γ . In turn, inserting (3.12), (3.15) and (7.20) into (7.23) implies (3.16)

$$D_\varepsilon[m] \stackrel{(7.23)}{\lesssim} \frac{t}{\sqrt{Q-1}} \int_{\mathbb{T}^2} |\nabla \bar{m}_3| \, d^2x.$$

Furthermore, Poincaré's inequality, (7.23), (3.12) and $\mu \sim 1$ yield (3.13)

$$\int_{\mathbb{T}^2 \times (0,1)} |m - \chi_{(0,1)} \bar{m}|^2 \, d^3x \lesssim \int_{\mathbb{T}^2 \times (0,1)} |\partial_3 m|^2 \, d^3x \stackrel{(7.23)}{\lesssim} t^3 \sqrt{Q-1} e^{-2\pi t^{-1}\sqrt{Q-1}}.$$

Finally, we deduce (3.14) from (7.22), (3.15) and (3.16)

$$\int_{\mathbb{T}^2 \times (0,1)} (m_1^2 + m_2^2) \, d^3x \stackrel{(7.22)}{\lesssim} \varepsilon \left(\int_{\mathbb{T}^2} |\nabla \bar{m}_3| \, d^2x + D_\varepsilon[m] \right) \lesssim e^{-2\pi t^{-1}\sqrt{Q-1}}, \quad (7.27)$$

which completes the proof. □

7.4 Proof of Theorem 3.4

Proof of Theorem 3.4. The proof is analogous to the proof of Theorem 3.7. □

Acknowledgements The work of CBM was supported, in part, by NSF via grants DMS-1313687 and DMS-1614948. FN thanks the New Jersey Institute of Technology for its hospitality during a visit in Newark and the Heidelberg Graduate School of Mathematical and Computational Methods for the Sciences for financial support. Furthermore, the authors thank Christof Melcher for stimulating questions that led to Remark 2.1 and Pierre Bousquet for pointing us to [28].

Appendix A

We give a proof for the continuity of $\varepsilon \mapsto \lambda_c(\varepsilon)$, the critical value of λ . Furthermore, we record a few well-known results that are used in the paper. For the convenience of the reader, we also give the proofs.

For $0 < \varepsilon < 1$, we define the critical value of λ where $\min F_{\varepsilon,\lambda}$ becomes negative as

$$\lambda_c(\varepsilon) := \inf\{\lambda : \min F_{\varepsilon,\lambda} < 0\}. \quad (\text{A.1})$$

Lemma A.1. *The function $\lambda_c : (0, 1) \rightarrow \mathbb{R}$ (see (A.1)) is Lipschitz-continuous on compact subsets of $(0, 1)$.*

Proof. The main idea is to express λ_c as the infimum over $\lambda_{c,m}$, where m is held fixed (see (A.3)) and to deduce regularity of λ_c from the regularity of $\lambda_{c,m}$. We define

$$X := \{m \in H^1(\mathbb{T}^2; \mathbb{S}^2) : m \text{ is not constant}\} \quad (\text{A.2})$$

and introduce, for any $m \in X$, the function

$$\lambda_{c,m} : (0, 1) \rightarrow \mathbb{R}, \quad \varepsilon \mapsto \lambda_{c,m}(\varepsilon) := \inf\{\lambda : F_{\varepsilon,\lambda}[m] < 0\}. \quad (\text{A.3})$$

Note that $F_{\varepsilon,\lambda}[m] \geq 0$ if m is constant and that $\lambda \mapsto F_{\varepsilon,\lambda}[m]$ is strictly monotone (for ε and $m \in X$ fixed). Hence, we may rewrite

$$\begin{aligned} \lambda_c(\varepsilon) &= \inf\{\lambda : \exists m \in X \text{ s.t. } F_{\varepsilon,\lambda}[m] < 0\} \\ &= \inf\{\lambda : \exists m \in X \text{ s.t. } \lambda > \lambda_{c,m}(\varepsilon)\} = \inf_{m \in X} \lambda_{c,m}(\varepsilon). \end{aligned} \quad (\text{A.4})$$

Step 1: Regularity of $\lambda_{c,m}$. We claim that

$$\left| \frac{d}{d\varepsilon} \lambda_{c,m}(\varepsilon) \right| \leq \left(1 + \frac{1}{|\log \varepsilon|} \right) \frac{\lambda_{c,m}(\varepsilon)}{\varepsilon} \quad \text{for all } m \in X. \quad (\text{A.5})$$

To prove (A.5), fix $m \in X$ and abbreviate

$$a = \int_{\mathbb{T}^2} |\nabla m|^2 dx, \quad b := \int_{\mathbb{T}^2} (1 - m_3^2) dx \quad c := \int_{\mathbb{T}^2} |\nabla^{1/2} m_3|^2 dx,$$

so that $F_{\varepsilon,\lambda}[m] = \frac{\varepsilon}{2}a + \frac{b}{2\varepsilon} - \frac{\lambda}{|\log \varepsilon|}c$ with partial derivatives

$$\partial_\varepsilon F_{\varepsilon,\lambda}[m] = \frac{a}{2} - \frac{b}{2\varepsilon^2} - \frac{\lambda c}{\varepsilon |\log \varepsilon|^2} \quad \text{and} \quad \partial_\lambda F_{\varepsilon,\lambda}[m] = -\frac{c}{|\log \varepsilon|}.$$

By continuity of $(\varepsilon, \lambda) \mapsto F_{\varepsilon,\lambda}[m]$ and strict monotonicity in λ , we deduce from (A.3) that $\lambda_{c,m}$ satisfies $F_{\varepsilon,\lambda_{c,m}(\varepsilon)}[m] = 0$ for all $\varepsilon \in (0, 1)$ and, furthermore, that it is the

only function with this property. Then the implicit function theorem asserts that $\lambda_{c,m}$ is $C^1((0, 1))$ with

$$\begin{aligned} \frac{d}{d\varepsilon} \lambda_{c,m}(\varepsilon) &= -(\partial_\lambda F_{\varepsilon,\lambda}[m])^{-1} \partial_\varepsilon F_{\varepsilon,\lambda}[m] \\ &= \frac{|\log \varepsilon|}{c} \left(\frac{a}{2} - \frac{b}{2\varepsilon^2} - \frac{\lambda c}{\varepsilon |\log \varepsilon|^2} \right). \end{aligned} \quad (\text{A.6})$$

Inserting the identity $F_{\varepsilon,\lambda_{c,m}(\varepsilon)}[m] = \frac{\varepsilon}{2}a + \frac{b}{2\varepsilon} - \frac{\lambda_{c,m}(\varepsilon)}{|\log \varepsilon|}c = 0$ into (A.6), we obtain the estimate

$$\left| \frac{d}{d\varepsilon} \lambda_{c,m}(\varepsilon) \right| \leq \frac{|\log \varepsilon|}{\varepsilon} \left(\frac{\frac{\varepsilon a}{2} + \frac{b}{2\varepsilon}}{c} \right) + \frac{\lambda_{c,m}(\varepsilon)}{\varepsilon |\log \varepsilon|} \leq \left(1 + \frac{1}{|\log \varepsilon|} \right) \frac{\lambda_{c,m}(\varepsilon)}{\varepsilon} \quad (\text{A.7})$$

which completes the proof of (A.5).

Step 2: Regularity of λ_c . The metric space $(X, \|\cdot\|_{H^1})$ is separable as a subset of the separable metric space $H^1(\mathbb{T}^2; \mathbb{R}^3)$ and hence there exists a dense countable subset $\{m_n : n \in \mathbb{N}\} \subset X$. Let $\delta \in (0, 1/2)$ and define $M := \sup_{\varepsilon \in [\delta, 1-\delta]} |\lambda_{c,m_1}(\varepsilon)| < +\infty$. Then the functions

$$g_n : [\delta, 1 - \delta] \rightarrow \mathbb{R}, \quad \varepsilon \mapsto g_n(\varepsilon) = \min\{\lambda_{c,m_n}(\varepsilon), M\} \quad (\text{A.8})$$

are Lipschitz-continuous for all $n \in \mathbb{N}$. Furthermore, by (A.5), their Lipschitz-constant is bounded by $\delta^{-1}(1 + \frac{1}{|\log \delta|})M$ (independent of $n \in \mathbb{N}$). Define the sequence of functions $f_k := \min_{1 \leq n \leq k} g_n$ and observe that

- (i) $\|f_k\|_{C^0([\delta, 1-\delta])} \leq M$ for all $k \in \mathbb{N}$,
- (ii) f_k is Lipschitz continuous with Lipschitz constant bounded by $\delta^{-1}(1 + \frac{1}{|\log \delta|})M$,
- (iii) $f_k(\varepsilon) \rightarrow \lambda_c(\varepsilon)$ as $k \rightarrow \infty$ for all $\varepsilon \in [\delta, 1 - \delta]$.

The last point follows from (A.4), the density of $\{m_n : n \in \mathbb{N}\} \subset X$ and continuity of $m \mapsto F_{\varepsilon,\lambda}[m]$. Now the compact embedding $C^{0,1}([\delta, 1 - \delta]) \hookrightarrow C^0([\delta, 1 - \delta])$ implies that $f_k \rightarrow f$ uniformly for some $f \in C^{0,1}([\delta, 1 - \delta])$ with Lipschitz constant bounded by $\delta^{-1}(1 + \frac{1}{|\log \delta|})M$. By uniqueness of the limit we conclude that $f = \lambda_c$, which completes the proof. \square

It is well-known that if $m \in H^1$ takes values in \mathbb{S}^2 , this implies certain estimates for the gradient ∇m (see, e.g., [40]). Since these estimates are used frequently throughout our paper, we record them in the following Lemma.

Lemma A.2. *Let $\Omega \subset \mathbb{R}^n$ be open and $m \in H^1(\Omega, \mathbb{S}^2)$. Then for every $\varepsilon > 0$ we have*

$$(i) \quad \frac{|\nabla m_3|^2}{1 - m_3^2} \leq |\nabla m|^2 \quad \text{for a.e. } x \in \Omega \text{ with } |m_3(x)| < 1, \quad (\text{A.9})$$

$$(ii) \quad |\nabla m_3| \leq \frac{\varepsilon}{2} |\nabla m|^2 + \frac{1 - m_3^2}{2\varepsilon} \quad \text{for a.e. } x \in \Omega. \quad (\text{A.10})$$

Proof. To prove (i), we apply the weak chain rule to the constraint $|m|^2 = 1$, which yields

$$-m_3 \nabla m_3 = m_1 \nabla m_1 + m_2 \nabla m_2$$

a.e. in Ω . After squaring both sides and applying the n -dimensional Cauchy-Schwarz inequality, we obtain

$$m_3^2 |\nabla m_3|^2 \leq (m_1^2 + m_2^2) (|\nabla m_1|^2 + |\nabla m_2|^2).$$

Finally we add $(m_1^2 + m_2^2) |\nabla m_3|^2$ to both sides. Since $|m|^2 = 1$, this yields

$$|\nabla m_3|^2 \leq (1 - m_3^2) |\nabla m|^2,$$

and hence proves (A.9).

We turn to the proof of (ii). Since $\nabla m_3 = 0$ almost everywhere on the set $\{x \in \Omega : |m_3(x)| = 1\}$, it remains to prove (A.10) on $\{x \in \Omega, |m_3(x)| < 1\}$. This follows from (A.9) upon an application of Young's inequality

$$2|\nabla m_3| \leq \frac{\varepsilon |\nabla m_3|^2}{1 - m_3^2} + \frac{1 - m_3^2}{\varepsilon} \stackrel{(A.9)}{\leq} \varepsilon |\nabla m|^2 + \frac{1 - m_3^2}{\varepsilon},$$

which concluded the proof. \square

In the following Lemma, we record a consequence of Jensen's inequality for the gradients of e_3 -averages.

Lemma A.3. *For every $p \in [1, \infty)$ and every $f \in W^{1,p}(\mathbb{T}^2 \times (0, 1))$, we have*

$$\int_{\mathbb{T}^2} \left| \nabla' \int_0^1 f(x', x_3) dx_3 \right|^p d^2 x' \leq \int_{\mathbb{T}^2 \times (0, 1)} |\nabla' f|^p d^3 x. \quad (\text{A.11})$$

Proof. Assume for a moment that $f \in C^\infty(\mathbb{T}^2 \times (0, 1))$. Since $|\cdot|^p : \mathbb{R}^2 \rightarrow \mathbb{R}$ (the p -th power of the euclidean norm) is a convex function, an application of Jensen's inequality (for two-dimensions) then yields

$$\left| \int_0^1 \nabla' f(x', x_3) dx_3 \right|^p d^2 x' \leq \int_0^1 |\nabla' f(x', x_3)|^p dx_3 \quad \text{for every } x' \in \mathbb{T}^2. \quad (\text{A.12})$$

For $f \in C^\infty(\mathbb{T}^2 \times (0, 1))$, we can change the order of integration and differentiation, so that (A.11) follows from (A.12) after integration over \mathbb{T}^2

$$\begin{aligned} \int_{\mathbb{T}^2} \left| \nabla' \int_0^1 f(x', x_3) dx_3 \right|^p d^2 x' &= \int_{\mathbb{T}^2} \left| \int_0^1 \nabla' f(x', x_3) dx_3 \right|^p d^2 x' \\ &\stackrel{(\text{A.12})}{\leq} \int_{\mathbb{T}^2} \int_0^1 |\nabla' f(x', x_3)|^p dx_3 d^2 x'. \end{aligned} \quad (\text{A.13})$$

Finally, (A.13) extends to any $f \in W^{1,p}(\mathbb{T}^2 \times (0,1))$ by a standard approximation argument using lower semi-continuity of the $W^{1,p}(\mathbb{T}^2)$ norm with respect to weak convergence of the e_3 -averages. \square

The next Lemma relates the real space formulation of the homogeneous $H^{1/2}$ -norm to its Fourier representation.

Lemma A.4. *For every smooth function $f : \mathbb{T}_\ell^2 \rightarrow \mathbb{R}$, the following holds*

$$\int_{\mathbb{T}_\ell^2} |\nabla^{1/2} f|^2 \, d^2x := \frac{1}{\ell^2} \sum_{k \in \frac{2\pi}{\ell} \mathbb{Z}^2} |k| |\widehat{f}_k|^2 = \frac{1}{4\pi} \int_{\mathbb{T}_\ell^2} \int_{\mathbb{R}^2} \frac{|f(x) - f(y)|^2}{|x - y|^3} \, d^2x \, d^2y. \quad (\text{A.14})$$

Proof. First we prove the identity

$$\int_{\mathbb{R}^2} |e^{ik \cdot x} - 1|^2 \frac{1}{|x|^3} \, d^2x = 4\pi |k| \quad \text{for every } k \in \frac{2\pi}{\ell} \mathbb{Z}^2. \quad (\text{A.15})$$

By scaling and rotational symmetry, we have

$$\int_{\mathbb{R}^2} |e^{ik \cdot x} - 1|^2 \frac{1}{|x|^3} \, d^2x = |k| \int_{\mathbb{R}^2} |e^{ix_1} - 1|^2 \frac{1}{|x|^3} \, d^2x. \quad (\text{A.16})$$

We evaluate the last integral in polar coordinates. On substituting $\rho = \frac{r \cos \theta}{2}$, we obtain

$$\begin{aligned} \int_{\mathbb{R}^2} |e^{ix_1} - 1|^2 \frac{1}{|x|^3} \, d^2x &= \int_{\mathbb{R}^2} |e^{\frac{ix_1}{2}} - e^{-\frac{ix_1}{2}}|^2 \frac{1}{|x|^3} \, d^2x = \int_0^{2\pi} \int_0^\infty 4 \sin^2 \left(\frac{r \cos \theta}{2} \right) \frac{1}{r^3} r \, d\theta \, dr \\ &= 2 \int_0^{2\pi} |\cos \theta| \, d\theta \int_0^\infty \frac{\sin^2 \rho}{\rho^2} \, d\rho = 4\pi. \end{aligned} \quad (\text{A.17})$$

Together, (A.16) and (A.17) prove (A.15).

With (A.15) at hand, we will now prove (A.14). By a variable transformation and Fubini's Theorem, we obtain

$$\int_{\mathbb{T}_\ell^2} \int_{\mathbb{R}^2} \frac{|f(x) - f(y)|^2}{|x - y|^3} \, d^2x \, d^2y = \int_{\mathbb{R}^2} \int_{\mathbb{T}_\ell^2} |f(z + y) - f(y)|^2 \, d^2y \frac{1}{|z|^3} \, d^2z.$$

Rewriting the inner integral in Fourier space and using Fubini's Theorem again yields

$$\begin{aligned} \int_{\mathbb{R}^2} \int_{\mathbb{T}_\ell^2} |f(z + y) - f(y)|^2 \, d^2y \frac{1}{|z|^3} \, d^2z &= \frac{1}{\ell^2} \int_{\mathbb{R}^2} \sum_{k \in \frac{2\pi}{\ell} \mathbb{Z}^2} |e^{-ik \cdot z} - 1|^2 |\widehat{f}_k|^2 \frac{1}{|z|^3} \, d^2z \\ &= \frac{1}{\ell^2} \sum_{k \in \frac{2\pi}{\ell} \mathbb{Z}^2} |\widehat{f}_k|^2 \int_{\mathbb{R}^2} |e^{-ik \cdot z} - 1|^2 \frac{1}{|z|^3} \, d^2z \stackrel{(\text{A.15})}{=} \frac{4\pi}{\ell^2} \sum_{k \in \frac{2\pi}{\ell} \mathbb{Z}^2} |k| |\widehat{f}_k|^2, \end{aligned}$$

which gives the desired formula. □

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