Traveling-wave solutions for a thin-film equation related to the spin coating process

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We study a thin-film problem related to the spin-coating process in which a fluid coats a rotating surface. Our interest lies in the contact-line region for which we propose a simplified traveling-wave approximation. We construct solutions to this problem by a shooting method that matches solution branches in the contact-line region and in the interior of the droplet. Furthermore, we observe qualitative properties of the solution connected to the fourth-order nature of the equation, such as a bump in the film height close to the contact line that is elevated from the average height of the film and can also be observed in experimental data.

Key Words: PDEs in connection with fluid mechanics; Traveling wave solutions; Nonlinear fourth-order equations; Existence and uniqueness; Thin fluid films.

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1 Introduction

In this paper, we investigate the axisymmetric thin-film equation

$$\partial_t h + r^{-1}\partial_r\left(r^2 m(h)h\right) + \epsilon r^{-1}\partial_r\left(rm(h)h\partial_r\left(r^{-1}\partial_r\left(r\partial_r h\right)\right)\right) = 0, \quad t > 0, \ \ r > 0. \quad (1.1)$$

Equation (1.1) models the height $h = h(t, r)$ of a viscous thin fluid film moving on a rotating substrate as a function of time t and radius r. It can be derived from the Navier-Stokes equations including the action of centrifugal force due to the rotation of the substrate. Here, $m(h) = \lambda h + h^2$ is called mobility, the real parameter $\lambda > 0$ is called slip length, and the parameter $\epsilon > 0$ determines the ratio between capillary and centrifugal forces, determined by the terms $\epsilon r^{-1}\partial_r\left(rm(h)h\partial_r\left(r^{-1}\partial_r\left(r\partial_r h\right)\right)\right)$ and $r^{-1}\partial_r\left(r^2 m(h)h\right)$, respectively. Note that $\epsilon r^{-1}\partial_r\left(rm(h)h\partial_r\left(r^{-1}\partial_r\left(r\partial_r h\right)\right)\right)$ can be seen as the analogue of a viscous regularizing term of the problem, despite being not a second but a fourth-order term. We refer to the appendix for details on the derivation and the dependence on physical parameters.

The motivation to study this problem comes from the spin-coating process, which is a mechanism used to apply thin films to substrates. In this procedure, the fluid is first deposited on the center of a substrate, the substrate is then rotated which leads to spreading of the fluid until the substrate is ultimately covered by the fluid uniformly. Spin coating processes have been an interesting subject in a variety of industrial applications such as photolithography used in semiconductor- and nanotechnology. At the same time the corresponding models have attracted interest in the physical and mathematical community. A related model to (1.1) for the spin-coating problem has been introduced in [11] where the authors investigated the rate of thinning of the flow. The model introduced in [11] has been modified through subsequent studies to include various factors such as thermal effects [28], the effect of the Coriolis force [23], air flow [21], air shear [20], surface tension [27], non-Newtonian fluids [1], topographic effects [19, 26], and evaporation [7].

In this paper, we consider a spreading viscous thin film on an axisymmetric rotating plate. We construct traveling-wave solutions to an ordinary differential equation (ODE), obtained by approximating (1.1), and analyze its properties. More precisely, in Section 2 we approximate the thin-film equation (1.1) to obtain the following ODE

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(Vh - m(h)h + m(h)h\frac{\mathrm{d}^3 h}{\mathrm{d}x^3}\right) = 0,$$

for traveling-wave solutions $h = h(x+Vt)$, where V denotes the speed of propagation. By an integration together with an assumption of the flow shape, the ODE can be reduced to a third order autonomous ODE. Due to the different dominant forces, we divide the analysis into two regions: the region where the flow spreads by a centrifugal force and the region where the flow is near the contact line, i.e., in contact with a solid. We use two directional shooting arguments and standard ODE theory in order to obtain solutions for two regions. Finally, we match the solutions in the intermediate region to find appropriate solutions having bumps as sketched in Fig. 1 and Fig. 2 in Section 2.

The spreading motion of liquid thin films depends on a force balance between viscosity, surface tension and further forces such as gravity. Corresponding thin-film type models have been intensively studied in the literature. We give some relevant examples: In [22], the authors solve a partial differential equation (PDE) for a thin liquid drop draining down a vertical wall by using an asymptotic method and matching solutions between inner region and outer region. The fourth-order PDE in [22] has been modeled with a force balance between gravity and surface tension. The authors in [3] also used an asymptotic method as well as a shooting method in order to obtain traveling-wave solutions. One can also find that in [5, 6], traveling-wave solutions play an important role in fourth-order PDEs describing a thin liquid film on an inclined substrate. Existence of traveling-wave solutions has been shown in [6] by means of a Lyapunov function for an ODE derived from the PDE employing a topological argument. In [5], the authors have investigated the stability of traveling-wave solutions by considering the Evans function for the ODE. We also would like to point that in the works [14, 24] the behavior of a retracting liquid has been analyzed using matched asymptotics. As in our work, the Navier-slip condition is used. In particular, the authors in [24] have investigated a wide range of slip length parameters for droplet dynamics.

The structure of the paper is as follows: In Section 2, we derive an ODE for traveling-wave solutions. Conditions of the flow shape are also introduced in this section. In Section 3 we present the main result on the existence and uniqueness of the solutions satisfying these conditions. The proof of existence is given in Sections 4–7 and the proof of the uniqueness is given in Section 8. In the Appendix, we give a derivation of the thin-film equation (1.1) in the lubrication approximation regime in the setting of the spin-coating process.

2 The model and traveling waves

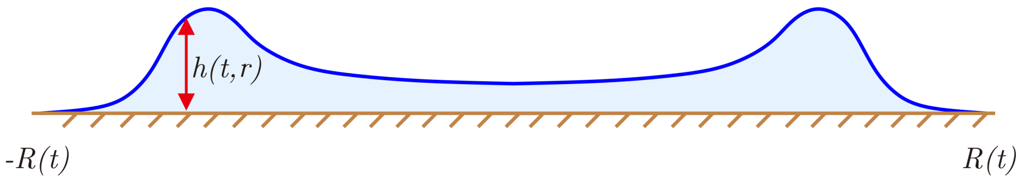

Figure 1. Geometry of the problem.

Let us assume the setting sketched in Fig. 1 where the film has extended to the point $r = R(t)$, the height of the film h is in good approximation a constant for $r < R$ and it perturbs around $h = 0$ in the region $|r - R|/R \ll 1$. Setting $r =: R - x$ and expanding (1.1) in powers of R^{-1}, we then formally obtain

$$\partial_t h - R\left(\partial_x(m(h)h) + O(R^{-1})\right) + \epsilon\left(\partial_x\left(m(h)h\partial_x^3 h\right) + O(R^{-1})\right) = 0, \quad t > 0,\ \ x \in \mathbb{R}. \tag{2.1}$$

We introduce new variables by

$$x =: \epsilon^{\frac{3}{8}} R^{-\frac{1}{2}} x_*, \qquad h =: \epsilon^{\frac{1}{8}} R^{-\frac{1}{2}} h_*, \qquad \lambda = \epsilon^{\frac{1}{8}} R^{-\frac{1}{2}} \lambda_*, \qquad t =: \epsilon^{\frac{1}{8}} R^{-\frac{1}{2}} t_*.$$

Neglecting higher order terms in R^{-1}, we then arrive at the problem

$$\partial_t h - \partial_x(m(h)h) + \partial_x\left(m(h)h\partial_x^3 h\right) = 0, \qquad t > 0,\ \ x \in \mathbb{R}, \tag{2.2}$$

where we have omitted the $*$ in our notation. The real parameter $\lambda > 0$ in the mobility $m(h) = \lambda h + h^2$ switches on and off the Navier-slip contribution, so that the addend λh should dominate if the film thickness h is below λ. In comparison to that, the term h^2 should play the dominant role in the bulk.

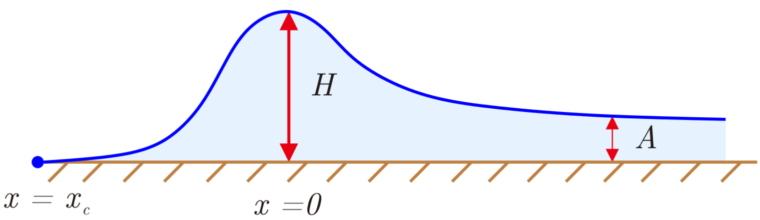

Figure 2. Geometry of the problem near the contact point on the left-hand side. The bump at $x = 0$ is related to the fact that equation (1.1) is of fourth order. For a corresponding second-order problem the profile would increase monotonically.

We investigate traveling-wave solutions to (2.2) by considering $h(t, x) := h_{\mathrm{TW}}(x + Vt)$, where $V \in \mathbb{R}$ is the speed of the wave. Skipping the index 'TW' and plugging this ansatz into (2.2), we obtain the ODE

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(Vh - m(h)h + m(h)h\frac{\mathrm{d}^3 h}{\mathrm{d}x^3}\right) = 0, \tag{2.3}$$

of the traveling-wave profile h (see Fig. 2). We will look for solutions of (2.3) which also satisfy the conditions

(A1) $h(x) \to A > 0$ as $x \to \infty$.

(A2) h attains its global maximum $H = h(0)$ at $x = 0$.

(A3) There exists a contact point $x_\mathrm{c} < 0$ such that $h(x_\mathrm{c}) = \frac{\mathrm{d}h}{\mathrm{d}x}(x_\mathrm{c}) = 0$ and such that $\operatorname{supp} \mathrm{h} = [x_\mathrm{c}, \infty)$.

The condition $h(x_\mathrm{c}) = 0$ simply determines the contact point x_c, whereas $\frac{\mathrm{d}h}{\mathrm{d}x}(x_\mathrm{c}) = 0$ ensures the contact angle to be zero (these boundary conditions are relevant in the case of complete wetting, see e.g. [10]). For given $A > 0$, the parameters H, V, and x_c are unknown a priori and have to be found as part of the solution. Instead of (A2), we will also use the slightly weaker condition

(A2*) $h(0) = H$, $\frac{\mathrm{d}h}{\mathrm{d}x}(0) = 0$, and $\frac{\mathrm{d}^2 h}{\mathrm{d}x^2}(0) = \kappa$,

where we have also introduced the new parameter $\kappa \leqslant 0$ to have a two-parameter family of solutions $h(x)$ depending on the free parameters H and κ.

We will show that the solution of this problem also satisfies two additional properties related to the velocity and the fluid profile near the contact point:

(B1) The velocity V is determined in terms of the film height A by

$$V = \lambda A + A^2. \tag{2.4}$$

(B2) For $0 \leqslant x \ll 1$ and with $\theta := \sqrt{\frac{8V}{3\lambda}}$, $\beta = \frac{1}{4}(\sqrt{13} - 1)$, the solution satisfies

$$h(x_\mathrm{c} + x) = \theta x^{\frac{3}{2}}\left(1 + v\left(x^{\frac{3}{2}}, x^{\beta}\right)\right), \tag{2.5}$$

where v is analytic in a neighborhood of $(y_1, y_2) = (0, 0)$ with $v(0, 0) = 0$.

The first identity is trivial and we will see that it follows by integrating equation (2.3) utilizing (A1)–(A3). The second condition is more intricate: Note that the leading-order behavior in (2.5) can be guessed by observing that

$$\psi(x) := \theta x^{\frac{3}{2}} \tag{2.6}$$

is a solution of the problem

$$\lambda \psi \frac{\mathrm{d}^3 \psi}{\mathrm{d}x^3} = -V \qquad \text{for } x > 0, \tag{2.7 a}$$

$$\psi = \frac{\mathrm{d}\psi}{\mathrm{d}x} = 0 \qquad \text{at } x = 0. \tag{2.7 b}$$

Hence, ψ also represents a traveling-wave solution for the thin-film equation with quadratic mobility, where the mobility $m(h)$ is replaced by its dominating contribution λh as $h \searrow 0$. This explains the leading behavior of the solution. For the structure of the correction term, we refer to Section 6.

3 Main result

The main result in this paper is the construction of a traveling-wave solution to the approximated thin-film equation (2.2) satisfying (A1)–(A3) and (B1)–(B2) as stated in Section 2:

Theorem 3.1 (Existence and uniqueness of traveling wave solution) *For every asymptotic film height $A > 0$, there exists a unique solution h of* (2.3) *satisfying* (A1)–(A3). *The parameters $H > A$, $V > 0$ and $x_c < 0$ are uniquely determined by A. Furthermore, the conditions* (B1)–(B2) *are satisfied.*

The traveling solution constructed in Theorem 3.1 describes the profile of the propagating liquid thin film in the later stages of the spin-coating process when the tangential curvature of the expanding liquid film is relatively small and can be neglected. Theorem 3.1 also yields a formula for the speed of propagation V in terms of the film height A in the bulk. Qualitatively, the traveling wave solution exhibits an approximately constant film height in the bulk (i.e., as $x \to \infty$), while it exhibits a local bump near the contact line where the maximal film height is attained. See also [14, 24] where a similar shape has been found for retracting films. Mathematically, it is related to the fact that the equation (1.1) is a fourth-order equation for which a comparison principle does not hold.

Note that, in the limit $\epsilon \to 0$, the propagation of the liquid film formally is described by the conservation law

$$\partial_t h + r^{-1}\partial_r \left(r^2 m(h)h\right) = 0, \qquad \text{for } t > 0, \quad r > 0. \tag{3.1}$$

Indeed, we expect that in the limit $\epsilon \to 0$, the solutions of (1.1) converge to a viscosity solution (also called entropy solution but not to be confused with the notion of entropy-weak solutions for the thin-film equation) of (3.1). The standard way to construct such entropy solutions is via a second order viscosity approximation. We note that fourth-order approximations of conservation laws have been studied (see e.g. [12, 15]) However, we have not found literature about regularizing terms as in (1.1) where the regularizing term is nonlinear, degenerate parabolic, and of fourth-order. It hence seems to be an interesting open question to prove convergence of solutions of (1.1) to an entropy solution of (3.1).

Note that, in the setting described in (2.1), the film height A is approximatively given by $1/R(t)$ where $R(t)$ is the radius of the expanding film. Hence, (2.4) yields a formula for the speed of propagation for the expanding thin film in terms of the average film height and thus also in terms of the radius. We remark that V does not depend on the regularization parameter $\epsilon > 0$. In fact, formula (2.4) can also be obtained by exploiting the Rankine-Hugoniot condition for the Burger's type equation (3.1) (cf. [13]).

For the proof of the theorem, we use the following strategy:

1. We perform a first trivial integration of (2.3), keeping the asymptotic behavior (A1) of our solution in mind. As a result, we obtain a new equation for h in Section 4 as a third-order ODE.
2. *Shooting from* 0 *to* ∞: By standard ODE theory, we construct a two parameter family of solutions h obeying $h(0) = H$, $\frac{\mathrm{d}h}{\mathrm{d}x}(0) = 0$, $\frac{\mathrm{d}^2h}{\mathrm{d}x^2}(0) = \kappa$, where $H > 0$ and

$\kappa \leqslant 0$ are free parameters. Then we choose $\kappa = \kappa_+(H)$ such that for every $H > A$ the asymptotic behavior (A1) is fulfilled (see Fig. 3).

3. *Shooting from* x_{c} *to* 0: We construct a one-parameter family of solutions in a right-neighborhood of $x = x_{\mathrm{c}}$ (similar to reference [16]) and match the conditions $h(0) = H$ and $\frac{\mathrm{d}^2 h}{\mathrm{d}x^2}(0) = \kappa_+(H)$ for some $H > A$.

4. *Uniqueness:* We prove uniqueness under the assumptions (A1)–(A3) by a method first used in [4] in the context of source-type self-similar solutions.

4 Simplification of the traveling-wave equation

We integrate equation (2.3) and get

$$Vh - m(h)h + m(h)h\frac{\mathrm{d}^3 h}{\mathrm{d}x^3} = C, \tag{4.1}$$

where $C \in \mathbb{R}$ is an integration constant. Therefore

$$\frac{\mathrm{d}^3 h}{\mathrm{d}x^3} = F(h), \tag{4.2}$$

where

$$F(h) := 1 - \frac{V}{m(h)} + \frac{C}{m(h)h} = 1 - \frac{V}{\lambda h + h^2} + \frac{C}{\lambda h^2 + h^3}. \tag{4.3}$$

If $C > 0$, condition (A3) in conjunction with (4.2) and (4.3) would imply $\frac{\mathrm{d}^3 h}{\mathrm{d}x^3}(x) \geqslant c(x - x_{\mathrm{c}})^{-2}$ as $x \searrow x_{\mathrm{c}}$ for some $c > 0$. Integrating twice, this would lead to a logarithmic divergence of $\frac{\mathrm{d}h}{\mathrm{d}x}$ as $x \searrow x_{\mathrm{c}}$, thus violating condition (A3). In the same way also $C < 0$ can be excluded, that is, we necessarily have $C = 0$.

Equations (4.2) and (4.3) with $C = 0$ and $h \to A$ as $x \to \infty$ (cf. (A1)) imply that

$$\frac{\mathrm{d}^3 h}{\mathrm{d}x^3} \to 1 - \frac{V}{m(A)} = 1 - \frac{V}{\lambda A + A^2} \qquad \text{as } x \to \infty. \tag{4.4}$$

As any non-vanishing value of $\lim_{x\to\infty} \frac{\mathrm{d}^3 h}{\mathrm{d}x^3}$ would violate condition (A1), we necessarily have that the speed of the wave is given by (2.4), i.e.,

$$V = m(A) = \lambda A + A^2,$$

and the function $F(h)$ simplifies to

$$F(h) = 1 - \frac{V}{\lambda h + h^2} = 1 - \frac{\lambda A + A^2}{\lambda h + h^2}. \tag{4.5}$$

For later use, we note that $F \in C^\infty((0,\infty))$ is strictly increasing with $\lim_{h\searrow 0} F(h) = -\infty$, $\lim_{h\to\infty} F(h) = 1$, and $F(A) = 0$.

5 Behavior as $x \to \infty$

In this section, we construct solutions h of formula (4.2) satisfying the conditions (A1) and (A2*) (see Fig. 3).

We introduce the notation $h = h_\kappa$ for the corresponding solution of (4.2) and (4.5) which satisfies condition (A2*). We have the following result:

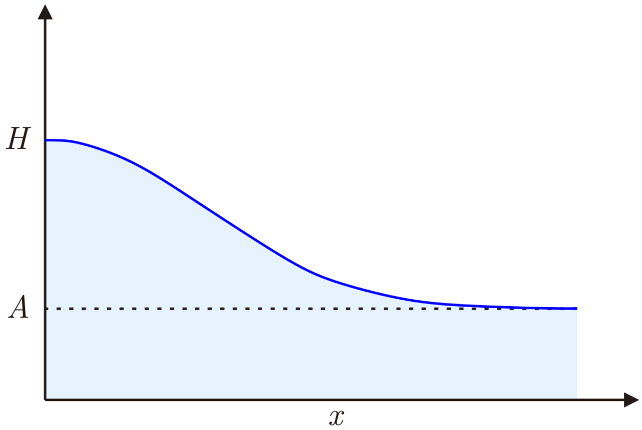

Figure 3. Sketch for solution satisfying (A1) and (A2*) in Section 5.

Proposition 5.1 (first shooting argument) *For any* $A > 0$ *and any* $H \geqslant A$ *there exists a unique* $\kappa = \kappa_+(H) \geqslant 0$ *such that the corresponding solution* h_{κ_+} *satisfies* (A1) *and* (A2*). *Furthermore,* h_{κ_+} *decreases monotonically in* $(0, \infty)$.

Proof of Proposition 5.1 If $H = A$ we have the trivial solution $h_0(x) \equiv A$ with $\kappa = 0$. In the following, we hence assume $H > A$. For given $\kappa \leqslant 0$, let $x_+ = x_+(\kappa) > 0$ be the smallest point such that $h_\kappa(x_+) = A$ if such a point exists and $x_+ = \infty$ otherwise. We define

$$\mathcal{K} := \Big\{ \kappa \in \mathbb{R} : x_+(\kappa) < \infty \Big\}. \tag{5.1}$$

Step 1. We claim that $\mathcal{K}$ is an open interval of the form

$$\mathcal{K} = (-\infty, \kappa_+) \tag{5.2}$$

for some $\kappa_+ := \kappa_+(H) \leqslant 0$. In order to see this, we first note that by standard ODE theory $x_+ = x_+(\kappa)$ depends continuously on the initial datum κ and hence $\mathcal{K}$ is an open set. By a Taylor expansion of $h_\kappa(x)$ around $x = 0$, we have

$$h_\kappa(x) = H + \frac{1}{2}\kappa x^2 + \frac{1}{6}F(h(\xi))x^3 \tag{5.3}$$

for any $x < x_+$ and for some $\xi \in (0, x)$. In view of (5.3) and since $0 < F(h) \leqslant 1$ for $h > A$, we conclude that $\kappa \in \mathcal{K}$ for κ sufficiently negative. In particular, $\mathcal{K} \neq \emptyset$.

Now, assume that $\kappa_1 \in \mathcal{K}$ and let $\kappa_2 < \kappa_1$. By construction, we have $h_{\kappa_i} \geqslant A$ in $[0, x_+]$ for $i = 1, 2$ where $x_+ := \min\{x_+(\kappa_1), x_+(\kappa_2)\}$. In view of the initial data at $x = 0$ and since F is monotonically increasing in h, this implies $h_{\kappa_2}(x) < h_{\kappa_1}(x)$ for all $x \in (0, x_+]$ and hence $A = h_{\kappa_2}(x_+) < h_{\kappa_1}(x_+)$. This shows that $x_+(\kappa_2) \leqslant x_+(\kappa_1) < \infty$ and hence $\kappa_2 \in \mathcal{K}$. It follows that $\mathcal{K} = (-\infty, \kappa_+)$ for some $\kappa_+ := \kappa_+(H) \in \mathbb{R}$.

For $H > A$ and $\kappa = 0$ we have $\frac{\mathrm{d}^2 h_0}{\mathrm{d}x^2}(0) = 0$. Since $\frac{\mathrm{d}^3 h}{\mathrm{d}x^3} = F(h) > 0$ for $h > A$, we have $\frac{\mathrm{d}^n h_0}{\mathrm{d}x^n}(x) \geqslant 0$ for all $n \in \mathbb{N}$ and $h_0 \nearrow \infty$ as $x \to \infty$. In particular $0 \notin \mathcal{K}$ and hence $\kappa_+ \leqslant 0$.

Step 2. We claim that h_{κ_+} is monotonically decreasing, i.e.,

$$\frac{\mathrm{d}h_{\kappa_+}}{\mathrm{d}x}(x) \leqslant 0 \qquad \text{for } x \in (0,\infty). \tag{5.4}$$

By the arguments in Step 1, $x_+(\kappa)$ is monotonically increasing in κ and $x_+(\kappa) \to \infty$ for $\kappa \nearrow \kappa_+$. Since also $\frac{\mathrm{d}h_\kappa}{\mathrm{d}x}(x)$ depends continuously on κ on compact subsets of $[0,\infty)$, in order to obtain (5.4), it is hence enough to show

$$\frac{\mathrm{d}h_{\kappa}}{\mathrm{d}x}(x) \leqslant 0 \qquad \text{for } x \in I := (0, x_+(\kappa)) \tag{5.5}$$

for any $\kappa \in \mathcal{K}$. Indeed, by (5.3) for any $\kappa \in \mathcal{K}$ we have $\frac{\mathrm{d}h_\kappa}{\mathrm{d}x}(x) < 0$ for $x \in (0,\eta)$ and $\eta > 0$ sufficiently small. Since $A \leqslant h_\kappa \leqslant H$ for $x \in I$, we have $\frac{\mathrm{d}^3 h_\kappa}{\mathrm{d}x^3} = F(h_\kappa) > 0$, i.e., $\frac{\mathrm{d}h_\kappa}{\mathrm{d}x}$ is a convex function in I. Further noting that $\frac{\mathrm{d}h_\kappa}{\mathrm{d}x}(x_+(\kappa)) \leqslant 0$ by construction, we necessarily have $\frac{\mathrm{d}h_\kappa}{\mathrm{d}x} < 0$ in I, i.e., (5.5) holds true.

Step 3. We have

$$h_{\kappa_+}(x) \to A \qquad \text{as } x \to \infty. \tag{5.6}$$

Indeed, by definition of $\mathcal{K}$ and since $\kappa_+ \notin \mathcal{K}$ we have $h_{\kappa_+}(x) > A$ for all $x \in (0,\infty)$. By Step 2, h is monotonically decreasing in $(0,\infty)$. In particular, there exists $C \geqslant A$ such that $h_{\kappa_+} \to C$ as $x \to \infty$. If $C > A$, we would have $\frac{\mathrm{d}^3 h_{\kappa_+}}{\mathrm{d}x^3}(x) = F(h_{\kappa_+}(x)) \geqslant F(C) > 0$ for all $x \in [0,\infty)$ which contradicts the fact that h decreases monotonically in $(0,\infty)$.

Step 4. It remains to show uniqueness. Arguing by contradiction, we assume that h_{κ_i} satisfies (A1) for $\kappa_2 < \kappa_1$. With the notation $\varphi := h_{\kappa_1} - h_{\kappa_2}$, we have $\varphi(0) = 0$, $\frac{\mathrm{d}\varphi}{\mathrm{d}x}(0) = 0$ and $\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2}(0) = \kappa_1 - \kappa_2 > 0$. It follows that $\varphi(x) > 0$ for $x \in (0,\eta)$ for some $\eta > 0$. In view of (4.2) and (4.5), we also have $\frac{\mathrm{d}^3\varphi}{\mathrm{d}x^3}(x) = F(h_{\kappa_1}(x)) - F(h_{\kappa_2}(x)) > 0$ for $x \in (0,\eta)$. Hence, $\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2}(x)$ is positive and increases strictly monotonically for $x > 0$. On the other hand, by assumption we have $\varphi(x) \to 0$ as $x \to \infty$, hence a contradiction. □

We conclude the section with a discussion of $\kappa_+(H)$.

Proposition 5.2 (Properties of $\kappa_+(H)$) *The function $\kappa_+ : [A,\infty) \to (-\infty, 0]$, defined in Proposition 5.1, has the following properties:*

(i) $\kappa_+(A) = 0$;

(ii) κ_+ *is strictly monotonically decreasing in* H;

(iii) $\kappa_+ \in C^0([A,\infty))$;

(iv) $\kappa_+(H) \to -\infty$ *as* $H \to \infty$.

Proof of Proposition 5.2 Let us denote by $h_{,H}$ the corresponding solution h of (4.2) and (4.5) which satisfies conditions (A1) and (A2*) with $H > 0$ and $\kappa = \kappa_+(H)$.

Proof of (i). This follows immediately, since $h(x) \equiv A$ solves (4.2) and (4.5) and satisfies (A1) and (A2*) for $\kappa = 0$.

Proof of (ii). For given $A \leqslant H_1 < H_2 \in \mathbb{R}$, we define $h_i := h_{,H_i}$ for $i = 1, 2$. By Proposition 5.1, we have in particular $h_i(x) > A$ for all $x \in (0,\infty)$. Arguing by contradiction, we assume $\kappa_+(H_2) \geqslant \kappa_+(H_1)$. With the notation $\varphi = h_2 - h_1$, we then have $\varphi(0) = H_2 - H_1 > 0$, $\frac{\mathrm{d}\varphi}{\mathrm{d}x}(0) = 0$ and $\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2}(0) = \kappa_+(H_2) - \kappa_+(H_1) \geqslant 0$. By (4.2) and (4.5), it then follows that $\varphi > 0$ and φ is monotonically increasing for all $x \in (0, \infty)$. This is a contradiction to the fact that by assumption we have $\lim_{x\to\infty} \varphi(x) = 0$.

Proof of (iii). By *(ii)*, κ_+ is a strictly decreasing function, and hence there can at most be a countable number of discontinuities. Arguing by contradiction, let us assume that κ_+ is discontinuous at $H_0 \geqslant A$. We first consider the case $H_0 > A$ and assume,

$$\delta_0 := \limsup_{\epsilon \searrow 0} \Big(\kappa_+(H_0 + \epsilon) - \kappa_+(H_0 - \epsilon) \Big) < 0.$$

For $\epsilon > 0$, let $\varphi_\epsilon := h_{,H_0+\epsilon} - h_{,H_0-\epsilon}$ and $\delta_\epsilon := \kappa_+(H_0 + \epsilon) - \kappa_+(H_0 - \epsilon) < 0$. We then have $\varphi_\epsilon(0) = 2\epsilon$, $\frac{\mathrm{d}\varphi_\epsilon}{\mathrm{d}x}(0) = 0$, $\frac{\mathrm{d}^2\varphi_\epsilon}{\mathrm{d}x^2}(0) = \delta_\epsilon$, $\frac{\mathrm{d}^3\varphi_\epsilon}{\mathrm{d}x^3}(x) = F(h_{,H_0+\epsilon}(x)) - F(h_{,H_0-\epsilon}(x))$. For $\epsilon > 0$ sufficiently small, we have by Taylor expansion for all $x \in (0, \infty)$

$$\varphi_\epsilon(x) \leqslant 2\epsilon + \frac{\delta_0}{4} x^2 + \frac{1}{6} \frac{\mathrm{d}^3\varphi_\epsilon}{\mathrm{d}x^3}(\xi) x^3 \qquad \text{for some } \xi \in (0, x), \tag{5.7}$$

which shows that for $\epsilon > 0$ sufficiently small there is $x^* > 0$ such that $\varphi_\epsilon(x^*) = 0$, $\frac{\mathrm{d}\varphi_\epsilon}{\mathrm{d}x}(x^*) < 0$, and $\frac{\mathrm{d}^2\varphi_\epsilon}{\mathrm{d}x^2}(x^*) < 0$. Since $\frac{\mathrm{d}^3\varphi_\epsilon}{\mathrm{d}x^3}$ has the same sign as φ_ϵ by equation (4.2) and the definition (4.5) of F, $\frac{\mathrm{d}^3\varphi_\epsilon}{\mathrm{d}x^3}(x) < 0$ for $x > x^*$, and hence φ_ϵ is negative and decreasing for all $x > x^*$. This contradicts the fact that by assumption, we have $\lim_{x\to\infty} \varphi_\epsilon(x) = 0$. A similar argument can be applied for $H_0 = A$ by defining $\varphi_\epsilon := h_{,A+\epsilon} - A$ and $\delta_\epsilon := \kappa_+(A + \epsilon) > 0$.

Proof of (iv). In view of *(ii)*, $\kappa_+(H)$ decreases monotonically. We assume by contradiction that $\kappa_+(H) \geqslant -K$ for all $H \geqslant A$ and for some $K > 0$. Then, by Proposition 5.1, $h := h_{,H}$ satisfies $h > A$, $\frac{\mathrm{d}^3 h}{\mathrm{d}x^3} > 0$ and hence $\frac{\mathrm{d}^2 h}{\mathrm{d}x^2} > -K$ in $(0, \infty)$. Since $\frac{\mathrm{d}h}{\mathrm{d}x}(0) = 0$, we get $\frac{\mathrm{d}h}{\mathrm{d}x} \geqslant -Kx$ and

$$h - A \geqslant (H - A) - \frac{1}{2} K x^2 \qquad \text{for } x \geqslant 0.$$

This implies $h - A \geqslant \frac{H-A}{2}$ for $x \leqslant \sqrt{\frac{H-A}{K}} =: x_K$ and thus $\frac{\mathrm{d}^3 h}{\mathrm{d}x^3} \geqslant \delta > 0$ for $x \leqslant x_K$ and $\delta = F\left(\frac{H+A}{2}\right) > 0$. In turn, this implies $\frac{\mathrm{d}^2 h}{\mathrm{d}x^2} \geqslant -K + \delta x$ and $\frac{\mathrm{d}h}{\mathrm{d}x} \geqslant -Kx + \frac{\delta}{2} x^2$ for $x \leqslant x_K$. If $\frac{H-A}{K} = x_K^2$ is sufficiently large, it follows that $h(x_K) > A$, $\frac{\mathrm{d}h}{\mathrm{d}x}(x_K) > 0$, and $\frac{\mathrm{d}^2 h}{\mathrm{d}x^2}(x_K) > 0$. In view of (4.2) and (4.5), this implies that h increases monotonically for $x > x_K$ and hence $h > A$ for all $x \geqslant 0$. This contradicts that, by construction, we have $\lim_{x\to\infty} h = A$. □

6 Behavior near the contact point

In this section, we construct a solution near the contact point x_{c} which satisfies (A3) and (B2) (see Fig. 2). In [17] as well as [2, 18] two approaches for the construction of solutions are detailed, one of which is based on invariant manifold theory for dynamical systems using the Hartman-Grobman theorem. Here we opt for the more direct approach in which we explicitly construct solutions by linearization and a fixed-point argument. For this, we shift equation (4.2) by $x \mapsto x - x_{\mathrm{c}}$ and take the assumption (A3) into account. Hence, we will consider

$$\frac{\mathrm{d}^3 h}{\mathrm{d}x^3} = F(h) \overset{(4.5)}{=} 1 - \frac{V}{\lambda h + h^2} = 1 - \frac{\lambda A + A^2}{\lambda h + h^2} \qquad \text{for } x > 0, \tag{6.1 a}$$

$$h = \frac{\mathrm{d}h}{\mathrm{d}x} = 0 \qquad \text{at } x = 0, \tag{6.1 b}$$

whose expected solution is sketched in Fig. 4.

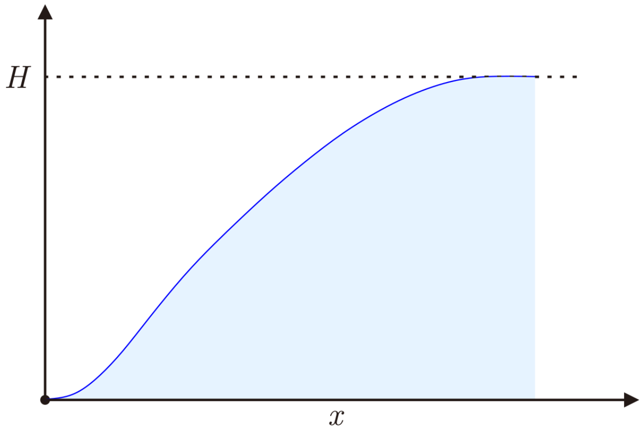

Figure 4. Sketch of the solution near the contact point $x = 0$.

New dependent variables

We factor off the leading-order traveling wave ψ (cf. (2.5)–(2.7)) by setting

$$h(x) =: \psi(x) S(x) \overset{(2.6)}{=} \theta x^{\frac{3}{2}} S(x) \tag{6.2}$$

for S to be determined and where θ is defined in (B2). We also define the scaling-invariant (logarithmic) derivative $D := x\frac{\mathrm{d}}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}\ln x}$. For later use, we note that

$$\frac{\mathrm{d}}{\mathrm{d}x} x^\mu = x^{\mu-1}(D+\mu) \qquad \text{for } \mu \in \mathbb{R}. \tag{6.3}$$

Also using the equivalent identity $Dx^\mu = x^\mu(D+\mu)$ for $\mu \in \mathbb{R}$, a straightforward calculation shows that

$$\frac{\mathrm{d}^3 h}{\mathrm{d}x^3} = x^{-1} D x^{-1} D x^{-1} D(\theta x^{\frac{3}{2}} S) = \theta x^{-\frac{3}{2}} q(D) S,$$

where the polynomial q is given by

$$q(\zeta) := \left(\zeta - \frac{1}{2}\right)\left(\zeta + \frac{1}{2}\right)\left(\zeta + \frac{3}{2}\right).$$

In terms of the scaling invariant derivative D, problem (6.1) can hence be expressed as

$$Sq(D)S + \frac{\theta}{\lambda} x^{\frac{3}{2}} S^2 q(D) S - \frac{1}{\lambda}(x^{\frac{3}{2}})^2 S^2 - \frac{1}{\theta} x^{\frac{3}{2}} S = -\frac{3}{8} \qquad \text{for } x > 0,$$

with the single boundary condition $S = 1$ at $x = 0$. In terms of the new variable u given by $S =: 1 + u$, we arrive at the problem

$$p(D)u = N(x^{\frac{3}{2}}, D, u) \qquad \text{for } x > 0, \tag{6.4 a}$$

$$u = 0 \qquad \text{at } x = 0, \tag{6.4 b}$$

where

$$p(\zeta) := q(\zeta) + q(0) = (\zeta + 1)(\zeta - \alpha)(\zeta - \beta), \tag{6.5}$$

$$\begin{aligned} N(x^{\frac{3}{2}}, D, u) := & -uq(D)u - \frac{\theta}{\lambda} x^{\frac{3}{2}} (1 + u)^2 q(D)(1 + u) \\ & + \frac{1}{\lambda}(x^{\frac{3}{2}})^2 (1 + u)^2 + \frac{1}{\theta} x^{\frac{3}{2}} (1 + u) \end{aligned} \tag{6.6}$$

with

$$\alpha := -\frac{1}{4}\left(\sqrt{13} + 1\right) < 0 \qquad \text{and} \qquad \beta := \frac{1}{4}\left(\sqrt{13} - 1\right) > 0. \tag{6.7}$$

Clearly, problem (6.4) does not allow for a solution u that is smooth up to the boundary $x = 0$, since terms containing the non-smooth factor $x^{\frac{3}{2}}$ appear in (6.6). In order to deal with this hurdle, we will apply an "unfolding of variables" in the sequel.

Unfolding of variables

We consider the following nonhomogeneous linearized problem:

$$p(D)u = g \qquad \text{for } x > 0, \tag{6.8 a}$$

$$u = 0 \qquad \text{at } x = 0. \tag{6.8 b}$$

The general solution of the initial value problem (6.8) is given as the sum of a particular solution and a linear combination of the solutions of the homogeneous equation, i.e.,

$$\ker p = \operatorname{span}\langle x^{-1}, x^{\alpha}, x^{\beta} \rangle.$$

Among these three solutions, the two solutions x^{-1} and x^{α} are ruled out by the boundary condition in (6.8). For this reason, and in view of (6.4), we expect the solution to be smooth in terms of the unfolding

$$u(x) = \overline{u}(x^{\frac{3}{2}}, bx^{\beta}),$$

where $\overline{u}(y_1, y_2)$ is a smooth function in (y_1, y_2) and $b \in \mathbb{R}$ is a free parameter. In fact, we will even show that $\overline{u}$ is real analytic. Correspondingly, as in [16, 17] we define

$$\overline{D} := \frac{3}{2} y_1 \partial_1 + \beta y_2 \partial_2.$$

In terms of the unfolded variables, we look for a solution $\overline{u}(y_1, y_2)$ of

$$p(\overline{D})\overline{u} = \overline{g} \qquad \text{for } y_1 \geqslant 0,\ y_2 \in \mathbb{R}, \tag{6.9 a}$$

$$(\overline{u}, \partial_2 \overline{u})\,(0, 0) = (0, -1) \tag{6.9 b}$$

for some smooth function $\overline{g}(y_1, y_2)$. Indeed, if $\overline{u}$ is a solution of (6.9), then by the chain rule it follows that $u(x) := \overline{u}(x^{\frac{3}{2}}, bx^{\beta})$ is a solution of (6.8) for every $b \in \mathbb{R}$. The nonlinear problem (6.4) can be correspondingly expressed in terms of unfolded variables as

$$p(\overline{D})\overline{u} = N(y_1, \overline{D}, \overline{u}) \qquad \text{for } y_1 \geqslant 0,\ y_2 \in \mathbb{R}, \tag{6.10 a}$$

$$(\overline{u}, \partial_2 \overline{u})\,(0,0) = (0,-1) \tag{6.10 b}$$

with

$$N(y_1, \overline{D}, \overline{u}) = -\overline{u} q(\overline{D})\overline{u} - \frac{\theta}{\lambda} y_1 (1+\overline{u})^2 q(\overline{D})(1+\bar{u}) + \frac{y_1^2}{\lambda}(1+\overline{u})^2 + \frac{y_1}{\theta}(1+\overline{u}). \tag{6.11}$$

Again, if $\overline{u}$ is a solution of (6.10), then a simple calculation shows that $u_b(x) := \overline{u}(x^{\frac{3}{2}}, bx^{\beta})$ is a solution of (6.4) for every $b \in \mathbb{R}$.

The linear problem

We first investigate the linear problem (6.9). In terms of $\overline{v}(y_1, y_2) := \overline{u}(y_1, y_2) + y_2$, we can as well consider the problem

$$p(\overline{D})\overline{v} = \overline{g} \qquad \text{for } y_1 \geqslant 0,\ y_2 \in \mathbb{R}, \tag{6.12 a}$$

$$(\overline{v}, \partial_2 \overline{v})\,(0,0) = (0,0), \tag{6.12 b}$$

since $p(\overline{D}) y_2 = 0$. Note that the boundary conditions in (6.12) imply corresponding boundary conditions, or compatibility conditions, for the right-hand side. Indeed, since $\overline{D}\overline{v}(0,0) = 0$ and by using the commutation relation $\partial_2 \overline{D} = (\overline{D} + \beta)\partial_2$, we have

$$p(\overline{D})\overline{v}(0,0) = p(0)\overline{v}(0,0) = 0,$$
$$\partial_2 p(\overline{D})\overline{v}(0,0) = p(\overline{D} + \beta)\partial_2 \overline{v}(0,0) = p(\beta)\partial_2 \overline{v}(0,0) = 0.$$

This calculation shows that the compatibility conditions $\overline{g}(0,0) = 0$ and $\partial_2 \overline{g}(0,0) = 0$ are necessary for the existence of a smooth solution.

The following lemma establishes existence and uniqueness for a solution of (6.12), if $\overline{g}$ satisfies appropriate compatibility conditions, and gives corresponding estimates for the solution operator. For the proof, we refer to [17, Prop. 1] where an operator of the same type is estimated by explicitly inverting $p\left(\overline{D}\right)$ using the method of characteristics.

Lemma 6.1 *Let $\ell_1, \ell_2 > 0$ and suppose that $\overline{g} \in C^\infty([0,\ell_1] \times [-\ell_2, \ell_2])$, $\overline{g} = \overline{g}(y_1, y_2)$, satisfies the compatibility conditions*

$$\overline{g}(0,0) = 0, \qquad \partial_2 \overline{g}(0,0) = 0.$$

Then problem (6.12) admits a solution $\overline{v} \in C^\infty([0,\ell_1] \times [-\ell_2, \ell_2])$. Furthermore, we have the maximal-regularity estimate

$$\sum_{m=0}^{3} \|\partial_1^k \partial_2^l \overline{D}^m \overline{v}\|_{C^0} \leqslant C \|\partial_1^k \partial_2^l \overline{g}\|_{C^0} \qquad \text{for all } (k,l) \in \mathbb{N}_0^2 \setminus \{(0,0),(0,1)\}, \tag{6.13}$$

where $\|\cdot\|_{C^0}$ denotes the supremum norm on $[0,\ell_1] \times [-\ell_2, \ell_2]$ and $C > 0$ is a universal constant. We denote the solution operator by T, i.e., $T\overline{g} := \overline{v}$.

Our aim is to estimate the solution of the nonlinear problem (6.10) by using the linear problem (6.12). In the following lemma, we define two norms and show that the solution of problem (6.12) can be estimated by the inhomogeneous term with the help of the norms following an argument in the proof of [17, Lemma 3].

Lemma 6.2 *Define the norms*

$$|||\overline{f}|||_0 := \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{\ell_1^k \ell_2^l}{k!l!} \|\partial_1^k \partial_2^l \overline{f}\|_{C^0} \qquad \text{and} \qquad |||\overline{f}|||_1 := \sum_{m=0}^{3} |||\overline{D}^m \overline{f}|||_0. \tag{6.14}$$

Then we have sub-multiplicativity of $|||\cdot|||_0$ *in the sense that*

$$|||\overline{f}\overline{g}|||_0 \leqslant |||\overline{f}|||_0 |||\overline{g}|||_0, \tag{6.15}$$

for $\overline{f}(y_1, y_2)$, $\overline{g}(y_1, y_2)$ *smooth. Furthermore, for* $\overline{g}$ *as in Lemma 6.1, we have*

$$|||T\overline{g}|||_1 \leqslant C|||\overline{g}|||_0,$$

where $T\overline{g}$ *is the solution of* (6.12) *and* $C > 0$ *is universal.*

Proof of Lemma 6.2 Suppose $\overline{f}(y_1, y_2)$ is smooth with $\overline{f}(0,0) = \partial_2 \overline{f}(0,0) = 0$, then

$$\|\overline{f}\|_{C^0} + \ell_2 \|\partial_2 \overline{f}\|_{C^0} \leqslant C\left(\ell_1 \|\partial_1 \overline{f}\|_{C^0} + \ell_2^2 \|\partial_2^2 \overline{f}\|_{C^0}\right), \tag{6.16}$$

with a universal $C > 0$, which as well as (6.15) is elementary to prove (cf. [17, proof of Lemma 3]). By means of Lemma 6.1 and (6.16), we obtain

$$|||T\overline{g}|||_1 = \sum_{m=0}^{3}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{\ell_1^k \ell_2^l}{k!l!} \|\partial_1^k \partial_2^l \overline{D}^m (T\overline{g})\|_{C^0} \leqslant C \sum_{(k,l)\neq(0,0),(0,1)} \frac{\ell_1^k \ell_2^l}{k!l!} \|\partial_1^k \partial_2^l \overline{g}\|_{C^0} \leqslant C|||\overline{g}|||_0,$$

where $C > 0$ is universal. □

The nonlinear problem

We proceed to the analysis of the nonlinear problem (6.10). We will apply a fixed-point argument to obtain existence of a local solution near the origin:

Proposition 6.3 *There is* $\ell_0 > 0$ *such that for* $\ell \in (0, \ell_0)$, *problem* (6.10) *with* $\ell_1 = \ell^2$, $\ell_2 = \ell$ *admits a solution* $\overline{u} \in C^\infty([0,\ell^2] \times [-\ell,\ell])$. *Furthermore, the solution* $\overline{u} = \overline{u}(y_1, y_2)$ *is analytic.*

Proof of Proposition 6.3 We first reduce problem (6.10) to the corresponding problem with homogeneous boundary conditions by setting $\overline{u}(y_1, y_2) =: \overline{v}(y_1, y_2) - y_2$. Then $\overline{v}(y_1, y_2)$ solves

$$p(\overline{D})\overline{v} = N(y_1, \overline{D}, \overline{v} - y_2) \qquad \text{for } y_1 > 0,\ y_2 \in \mathbb{R}, \tag{6.17 a}$$

$$(\overline{v}, \partial_2 \overline{v})(0,0) = (0,0). \tag{6.17 b}$$

We note that if $\overline{v} \in C^\infty([0,\ell^2] \times [-\ell,\ell])$ satisfies $\overline{v}(0,0) = \partial_2 \overline{v}(0,0) = 0$, then

$$\overline{g} := N(y_1, \bar{D}, \overline{v} - y_2)$$

satisfies $\overline{g} \in C^\infty([0,\ell^2]\times[-\ell,\ell])$ and $\overline{g}(0,0) = \partial_2\overline{g}(0,0) = 0$. This follows by direct computation, using the commutation relation $\partial_2\overline{D} = (\overline{D}+\beta)\partial_2$. By application of the solution operator T, constructed in Lemma 6.1, we hence obtain the fixed-point equation

$$\overline{v} = \mathcal{T}(\overline{v}), \qquad \text{where} \qquad \mathcal{T}(\overline{v}) := T\overline{g} = TN(y_1,\overline{D},\overline{v}-y_2). \tag{6.18}$$

We also define the metric space S by

$$\begin{aligned} S \ :=\ & \text{closure with respect to } |||\cdot|||_1 \text{ of} \\ & \left\{\overline{v}\in C^\infty([0,\ell^2]\times[-\ell,\ell]) \ :\ |||\overline{v}|||_1 \leqslant \ell,\, \overline{v}(0,0) = \partial_2\overline{v}(0,0) = 0\right\}, \end{aligned}$$

where the norms defined in (6.14) of Lemma 6.2. We claim that for an $\ell > 0$ sufficiently small, the fixed-point problem (6.18) allows for a unique solution in S. Our argument is based on Banach's fixed-point theorem.

By Lemma 6.2, we have

$$|||\mathcal{T}(\overline{v})|||_1 \leqslant C|||N(y_1,\overline{D},\overline{v}-y_2)|||_0, \tag{6.19}$$

$$|||\mathcal{T}(\overline{v}_1) - T(\overline{v}_2)|||_1 \leqslant C|||N(y_1,\overline{D},\overline{v}_1-y_2) - N(y_1,\overline{D},\overline{v}_2-y_2)|||_0 \tag{6.20}$$

for smooth functions $\overline{v}, \overline{v}_i$ $(i=1,2)$ satisfying the compatibility conditions (6.12 b), where here and in what follows in this proof, $C,\tilde{C},\hat{C} > 0$ are universal. Clearly, we have $|||q(\overline{D})y_2|||_0 \leqslant C\ell$. In addition, it is also easy to see that we have (cf. (6.11))

$$\begin{aligned} |||N(y_1,\overline{D},\overline{v}-y_2)|||_0 \leqslant{}& C\left(|||\overline{v}-y_2|||_0 + |||y_1(1+\overline{v}-y_2)^2|||_0\right)|||q(\overline{D})(1+\overline{v}-y_2)|||_0 \\ &+C|||y_1^2|||_0|||1+\overline{v}-y_2|||_0^2 + C|||y_1|||_0|||1+\overline{v}-y_2|||_0 \leqslant \tilde{C}\ell \end{aligned} \tag{6.21}$$

for $\overline{v}\in S$ and

$$\begin{aligned} &|||N(y_1,\overline{D},\overline{v}_1-y_2) - N(y_1,\overline{D},\overline{v}_2-y_2)|||_0 \\ &\leqslant C|||(\overline{v}_1-y_2)q(\overline{D})(\overline{v}_1-y_2) - (\overline{v}_2-y_2)q(\overline{D})(\overline{v}_2-y_2)|||_0 \\ &\quad +C|||y_1(1+(\overline{v}_1-y_2))^2q(\overline{D})(\overline{v}_1-y_2) - y_1(1+(\overline{v}_2-y_2))^2q(\overline{D})(\overline{v}_2-y_2)|||_0 \\ &\quad +C|||y_1^2|||_0|||(1+(\overline{v}_1-y_2))^2 - (1+(\overline{v}_2-y_2))^2|||_0 + C|||y_1|||_0|||\overline{v}_1-\overline{v}_2|||_0 \\ &\leqslant \tilde{C}\left(\ell + |||\overline{v}_1|||_1 + |||\overline{v}_2|||_1\right)|||\overline{v}_1-\overline{v}_2|||_1 \leqslant \hat{C}\ell|||\overline{v}_1-\overline{v}_2|||_1 \end{aligned} \tag{6.22}$$

for $\overline{v}_1,\overline{v}_2 \in S$. Using (6.20) and (6.22), by Banach's fixed-point theorem we obtain a unique solution of (6.18) in S for sufficiently small $\ell > 0$. □

We are ready to prove the existence of a solution of (6.1):

Proposition 6.4 *For any $b\in\mathbb{R}$ there is a unique solution h_b of* (6.1) *such that*

$$h_b(x) = \theta x^{\frac{3}{2}}\left(1+u_b(x)\right) \qquad \textit{for } 0\leqslant x\leqslant x_b^* := \min\left\{\ell^{\frac{4}{3}}, \left(\tfrac{\ell}{|b|}\right)^{\frac{1}{\beta}}\right\}, \tag{6.23}$$

where $u_b(x) = \overline{u}(x^{\frac{3}{2}}, bx^\beta)$ for some analytic function $\overline{u} : [0,\ell^2]\times[-\ell,\ell]\to\mathbb{R}$ with $\overline{u}(0,0) = 0$, $\partial_2\overline{u}(0,0) = -1$.

Proof of Proposition 6.4 Let $\overline{u}$ be the solution of Proposition 6.3. Then h_b, given by (6.23), satisfies (6.4). Furthermore, the value $x_b^* = \min\left\{\ell^{\frac{4}{3}}, \left(\frac{\ell}{|b|}\right)^{\frac{1}{\beta}}\right\}$ follows from the corresponding range of an analytic function $\overline{u}$ defined on $[0,\ell^2]\times[-\ell,\ell]$. □

7 Matching argument

In this section we will match the solution (6.23) obtained in Section 6 to the conditions (A1) and (A2*), described in Section 5. We use the same coordinates as in the previous section, i.e., the contact point is shifted to $x = 0$ (cf. (6.1)).

We first investigate the behavior of the solution in Proposition 6.4 in dependence of the parameter b. This will be used later for the matching argument.

Lemma 7.1 *Let $h_b(x)$ be the solution of* (6.23)*[should be (6.1)?], given in Proposition 6.4, and $\psi = \psi(x) = \theta x^{\frac{3}{2}}$ be the leading-order behavior of h_b, given in* (2.6)*. Let x_b be the maximal value such that h_b is smooth and satisfies $h_b > 0$ in $(0, x_b)$. Then $\frac{\mathrm{d}^k h_b}{\mathrm{d}x^k}(x)$ and $\frac{\mathrm{d}}{\mathrm{d}b}\frac{\mathrm{d}^k h_b}{\mathrm{d}x^k}(x)$ depend smoothly on b on compact subsets of $[0, x_b)$. Furthermore, we have*
(i) for any $x > 0$ and $b \leqslant 0$, we have for $k = 0, 1, 2, 3$

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k}(h_b - \psi)(x) \geqslant 0 \qquad \textit{(Overshooting)};$$

(ii) for any $x > 0$, we have for $k = 0, 1, 2, 3$

$$\frac{\mathrm{d}}{\mathrm{d}b}\frac{\mathrm{d}^k}{\mathrm{d}x^k}(h_b - \psi)(x) = \frac{\mathrm{d}}{\mathrm{d}b}\frac{\mathrm{d}^k}{\mathrm{d}x^k}h_b(x) < 0 \qquad \textit{(Monotonicity in } b\textit{)};$$

(iii) for any $b \in \mathbb{R}$, we have $x_b > 0$ and

$$x_b \searrow 0 \qquad \text{as } b \nearrow \infty \qquad \textit{(Undershooting)}.$$

Proof of Lemma 7.1 For the proof, we use representation (6.23) which gives precise notion of the behavior of $h_b(x)$ for $x \geqslant 0$ sufficiently small. Furthermore, we note that

$$\frac{\mathrm{d}}{\mathrm{d}b}u_b(x) = x^\beta \partial_2 \overline{u}\left(x^{\frac{3}{2}}, bx^\beta\right). \tag{7.1}$$

By (6.23), (7.1), and in view of (2.6) and (6.3), we have

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left(h_b(x) - \psi(x)\right) = -b\theta\left(\frac{\mathrm{d}^k}{\mathrm{d}x^k}x^{\frac{3}{2}+\beta}\right)(1 + o(1)), \tag{7.2 a}$$

$$\frac{\mathrm{d}}{\mathrm{d}b}\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left(h_b(x) - \psi(x)\right) = -\theta\left(\frac{\mathrm{d}^k}{\mathrm{d}x^k}x^{\frac{3}{2}+\beta}\right)(1 + o(1)), \tag{7.2 b}$$

for $k = 0, 1, 2$ and $x > 0$ sufficiently small. The continuous dependence of h_b and its derivatives follows by expansions (7.2) and standard ODE theory for larger values of x. Now we turn our attention to the proofs for assertions *(i)*, *(ii)*, and *(iii)*.

Proof of (i). By (7.2 a) and if $b < 0$, the estimate holds for $k = 0, 1, 2$ and for $x > 0$ sufficiently small. For $k = 3$, we use (2.7) and (6.1), i.e.,

$$\frac{\mathrm{d}^3}{\mathrm{d}x^3}\left(h_b(x) - \psi(x)\right) = 1 + V\frac{\lambda h_b + h_b^2 - \lambda\psi}{\lambda\psi(\lambda h_b + h_b^2)} \tag{7.3}$$

which is nonnegative if $h_b - \psi \geqslant 0$. Hence the estimate holds for $x > 0$ sufficiently small. For larger values of x, the estimate also follows by (7.3) and an ODE argument (in fact with a strict inequality). For $b = 0$, assertion *(i)* follows from the smooth dependence of the solution on b.

Proof of (ii). For $x > 0$ sufficiently small, the estimate holds by (7.2 b). For larger x, the estimate continues to hold by an ODE argument as in *(i)*, employing (7.3).

Proof of (iii). This claim follows by a Taylor expansion of $h_b(x)-\psi(x)$ at x_b^* (cf. (6.23)) up to second order, appropriately estimating the third derivative from above by using (7.3). This argument is detailed in a similar case in [17, proof of Lemma 5c]. □

With the help of Lemma 7.1 we can prove:

Lemma 7.2 *There exists $\hat{b} \geqslant 0$ maximal such that the solution $h_{\hat{b}}$, constructed in Proposition 6.4, meets the conditions*

$$h_{\hat{b}}(x) = A, \qquad \frac{\mathrm{d}h_{\hat{b}}}{\mathrm{d}x}(x) = 0, \qquad \text{and} \qquad \frac{\mathrm{d}^2 h_{\hat{b}}}{\mathrm{d}x^2}(x) < 0 \tag{7.4}$$

for some $x > 0$. We denote by $\hat{x}$ the minimal x such that (7.4) *holds true.*

Proof of Lemma 7.2 For any $b \in \mathbb{R}$, we define $\check{x}_b > 0$ as the minimal point with $h_b(\check{x}_b) = A$, if it exists and otherwise $\check{x}_b = \infty$. Furthermore, we set

$$\mathcal{B} := \{b \in \mathbb{R} : \check{x}_b < \infty\}.$$

By Lemma 7.1*(i)*, we have $(-\infty, 0] \subset \mathcal{B}$. In view of Lemma 7.1*(ii)*, *(iii)*, the set $\mathcal{B}$ is bounded from above. Lemma 7.1*(ii)* also shows that h_b is continuous and monotonically decreasing in b and hence $\mathcal{B} = (-\infty, \hat{b}]$ for some $\hat{b} \in \mathcal{B}$ with $\hat{b} \geqslant 0$. Then $\hat{x} = \check{x}_{\hat{b}}$ satisfies $h_{\hat{b}}(\hat{x}) = A$. We claim that $\frac{\mathrm{d}h_{\hat{b}}}{\mathrm{d}x}(\hat{x}) = 0$. Indeed, the inequality $\frac{\mathrm{d}h_{\hat{b}}}{\mathrm{d}x}(\hat{x}) > 0$ would contradict the fact that $\hat{b}$ is the supremum of $\mathcal{B}$ and $\frac{\mathrm{d}h_{\hat{b}}}{\mathrm{d}x}(\hat{x}) < 0$ would contradict the minimality of $\hat{x}$. By equation (6.1 a), $\frac{\mathrm{d}^2 h_{\hat{b}}}{\mathrm{d}x^2}$ is monotonically decreasing as long as $h_{\hat{b}} \leqslant A$. Since $\frac{\mathrm{d}h_{\hat{b}}}{\mathrm{d}x}(x) > 0$ for $x > 0$ sufficiently small, by (7.2 a) and $\frac{\mathrm{d}h_{\hat{b}}}{\mathrm{d}x}(\hat{x}) = 0$ we necessarily also have $\frac{\mathrm{d}^2 h_{\hat{b}}}{\mathrm{d}x^2}(\hat{x}) < 0$. □

We next discuss the set of heights H which represent the maximum of h_b for some b:

Lemma 7.3 *For any $H \geqslant A$, let $\check{x}_H > 0$ be such that*

$$h_{\check{b}} < H \text{ in } [0, \check{x}_H), \qquad h_{\check{b}}(\check{x}_H) = H, \qquad \frac{\mathrm{d}h_{\check{b}}}{\mathrm{d}x}(\check{x}_H) = 0, \qquad \frac{\mathrm{d}^2 h_{\check{b}}}{\mathrm{d}x^2}(\check{x}_H) < 0 \tag{7.5}$$

for some $\check{b} < \hat{b}$ if such a point exists and let

$$\mathcal{H} = \left\{H > A : \text{there are } \check{b} < \hat{b},\ \check{x}_H > 0 \text{ such that conditions (7.5) hold}\right\}.$$

Then there is $H_- > A$ such that $\mathcal{H} = (A, H_-)$. Furthermore, $\check{b} = \check{b}(H)$ is uniquely determined, monotonically decreasing in H and $\check{x}_H$ is monotonically increasing in H. Both functions are continuous in H.

Proof of Lemma 7.3 By Lemma 7.2 there is $\hat{b} \geqslant 0$ and $\hat{x} > 0$ such that the corresponding solution $h_{\hat{b}}$ satisfies $h_{\hat{b}} < A$ in $[0, \hat{x})$, $h_{\hat{b}}(\hat{x}) = A$, $\frac{\mathrm{d}h_{\hat{b}}}{\mathrm{d}x}(\hat{x}) = 0$ and $\frac{\mathrm{d}^2 h_{\hat{b}}}{\mathrm{d}x^2}(\hat{x}) < 0$. By Lemma 7.1, for $b < \hat{b}$ we have $\frac{\mathrm{d}h_b}{\mathrm{d}x}(\hat{x}) > 0$. Since by equation (6.1 a) we have $\frac{\mathrm{d}^3 h_b}{\mathrm{d}x^3}(x) \leqslant 1$, a Taylor expansion yields

$$\frac{\mathrm{d}h_b}{\mathrm{d}x}(x) \leqslant \frac{\mathrm{d}h_b}{\mathrm{d}x}(\hat{x}) + \frac{\mathrm{d}^2 h_b}{\mathrm{d}x^2}(\hat{x})(x-\hat{x}) + \frac{1}{2}(x-\hat{x})^2. \tag{7.6}$$

By Lemma 7.1, $\frac{\mathrm{d}^2 h_b}{\mathrm{d}x^2}(\hat{x})$ depends smoothly on b. In particular, we have $\frac{\mathrm{d}^2 h_b}{\mathrm{d}x^2}(\hat{x}) < 0$ and $|\frac{\mathrm{d}^2 h_b}{\mathrm{d}x^2}(\hat{x})|^2 \geqslant 2|\frac{\mathrm{d}h_b}{\mathrm{d}x}(\hat{x})|$ for $0 < \hat{b} - b \ll 1$. Since the quadratic right-hand side in (7.6) has a real solution if its discriminant is non-negative, by (7.6) and since $\frac{\mathrm{d}h_b}{\mathrm{d}x}$ is continuous, this implies that there is a minimal $\tilde{x}_b > \hat{x}$ such that $\frac{\mathrm{d}h_b}{\mathrm{d}x}(\tilde{x}_b) = 0$ and $\frac{\mathrm{d}^2 h_b}{\mathrm{d}x^2}(\tilde{x}_b) < 0$ for any $b \in (\hat{b} - \eta, \hat{b})$ and $\eta > 0$ sufficiently small. We also set $H_b := h_b(\tilde{x}_b)$. By Lemma 7.1 and the implicit function theorem, $\tilde{x}_b$ and H_b depend continuously on b. This shows that $(A, H_{\hat{b}-\eta}) \subset \mathcal{H}$ and hence $\mathcal{H} \neq \emptyset$. Using the same argument as before, based on the implicit function theorem, it follows that $\mathcal{H}$ is an open set. By Lemma 7.1*(i)*, it follows that $\mathcal{H}$ is a bounded set.

Let H_1, $H_2 \in \mathcal{H}$ with $H_1 < H_2$, $x_i := \check{x}_{H_i}$, $b_i := \check{b}(H_i)$, and $h_i := h_{b_i}$ for $i = 1, 2$. Lemma 7.1*(ii)* implies that $x_1 < x_2$ and $b_1 > b_2$. Together with Lemma 7.1 and the implicit function theorem, it also follows that $\check{x}_H$ and $\check{b}(H)$ are continuous functions in H.

It remains to show that $\mathcal{H}$ is connected. By Lemma 7.1*(ii)*, we have for $b \in (b_2, b_1)$ that $\frac{\mathrm{d}h_b}{\mathrm{d}x}(x_1) > \frac{\mathrm{d}h_1}{\mathrm{d}x}(x_1) = 0$ and $\frac{\mathrm{d}h_b}{\mathrm{d}x}(x_2) < \frac{\mathrm{d}h_2}{\mathrm{d}x}(x_2) = 0$. This implies that h_b has at least one maximum in (x_1, x_2) and we again denote by $\tilde{x}_b$ the minimal value for which a maximum is attained. Suppose $\frac{\mathrm{d}^2 h_b}{\mathrm{d}x^2}(\tilde{x}_b) = 0$. Because of $\frac{\mathrm{d}h_b}{\mathrm{d}x}(\tilde{x}_b) = 0$ and $h_b(\tilde{x}_b) > h_b(x_1) > h_1(x_1) = H_1 \geqslant A$ by Lemma 7.1*(ii)*, equation (6.1 *a*) implies that h_b is strictly increasing for $x \geqslant \tilde{x}_b$, which is a contradiction to h_b having a maximum at $x = \tilde{x}_b$. Thus by the implicit function theorem, $\tilde{x}_b$ depends continuously on b such that $\tilde{x}_b \to x_i$ as $b \to b_i$. Since $H_b = h_b(\tilde{x}_b)$ depends continuously on b, we get $[H_1, H_2] \subset \mathcal{H}$. □

We are now able to conclude:

Proposition 7.4 *For $H \in \mathcal{H} = (A, H_-)$, let*

$$\kappa_- : \mathcal{H} \to \mathbb{R}, \qquad \kappa_-(H) := \frac{\mathrm{d}^2 h_{\check{b}(H)}}{\mathrm{d}x^2}(\check{x}_H). \tag{7.7}$$

Then $\kappa_- \in C^0(\mathcal{H})$ and
(i) $-C \leqslant \kappa_- < 0$ for some universal $C > 0$.
(ii) $\kappa_-(H) \to 0$ as $H \to H_-$.

We are now ready to establish the existence part of Theorem 3.1:

Proposition 7.5 (Existence) *For any $A > 0$ there are $H > A$ and $x_{\mathrm{c}} < 0$ such that there exists a solution of* (2.3) *satisfying* (A1)–(A3) *and* (B1)–(B2).

Proof of Proposition 7.5 In view Proposition 5.2 and Proposition 7.4, the functions

$\kappa_+(H)$ and $\kappa_-(H)$ intersect in a point $H > A$. The corresponding solution $h_{,H}$ is the sought-after solution, whence the existence part of Theorem 3.1 follows. □

Proof of Proposition 7.4 The continuity of $\kappa_-(H)$ follows from Lemma 7.1 and the continuity of $\check{x}_H$ and $\check{b}(H)$. In the proof of Lemma 7.3, we already have shown that $\kappa_-(H) < 0$ for all $H \in \mathcal{H}$. Let $\tilde{H} \in \mathcal{H}$, let $\tilde{x} = \check{x}_{\tilde{H}}$, $\tilde{b} = \check{b}(\tilde{H})$ and let $\tilde{h} = h_{\tilde{b}}$. By Lemma 7.3 we have $\frac{\mathrm{d}^2\tilde{h}}{\mathrm{d}x^2}(\tilde{x}) > \frac{\mathrm{d}^2\tilde{h}}{\mathrm{d}x^2}(\hat{x}) > \frac{\mathrm{d}^2 h_{\hat{b}}}{\mathrm{d}x^2}(\hat{x}) > -\infty$, where we also used that $\tilde{h} > A$ in $(\hat{x}, \tilde{x})$ and hence $\frac{\mathrm{d}^3\tilde{h}}{\mathrm{d}x^3} > 0$ in the same interval by (4.2) and (4.5). This shows *(i)*. The arguments in the proof of Lemma 7.3 show that every point $H \in \mathcal{H}$ such that $\kappa_-(H) < 0$ is an interior point of $\mathcal{H}$. Together with *(i)*, this shows that *(ii)* holds. □

8 Uniqueness

In this section, we give an argument why any solution satisfying (A1)–(A3) is already unique (without the assumption (B2)). The method is based on an argument by Bernis, Peletier, and Williams in [4] for the proof of uniqueness of source-type self-similar solutions and has also been applied to other situations, e.g. [2, 9].

Proposition 8.1 (Uniqueness) *Suppose that $A > 0$. Then there exists at most one solution of* (2.3) *satisfying* (A1)–(A3).

Proof of Proposition 8.1 Let us suppose that there are two solutions h_i, $i = 1, 2$, of (2.3) satisfying (A1) and (A3). By a shift in x, we may assume that the left boundaries of the support of the functions h_i coincide at the same point x_c, i.e. $h_i \in C^1([x_\mathrm{c}, \infty)) \cap C^3((x_\mathrm{c}, \infty))$ with $h_i(x_\mathrm{c}) = \frac{\mathrm{d}h_i}{\mathrm{d}x}(x_\mathrm{c}) = 0$. We set $\varphi := h_1 - h_2$. We claim that it suffices to show that

$$\varphi \frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} \geqslant \frac{1}{2}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right)^2 \geqslant 0 \qquad \text{for all } x \geqslant x_\mathrm{c}. \tag{8.1}$$

Indeed, if (8.1) holds, then we also have

$$\frac{\mathrm{d}^2\varphi^2}{\mathrm{d}x^2} = 2\varphi\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} + 2\left(\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right)^2 \overset{(8.1)}{\geqslant} 3\left(\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right)^2 \geqslant 0.$$

The non-negative function φ^2 is therefore convex and because of (A1) and (A3) fulfills $\varphi^2(x_\mathrm{c}) = 0$ and $\varphi^2(x) \to 0$ as $x \to \infty$. This implies $\varphi^2 \equiv 0$ and hence $h_1 \equiv h_2$. In particular (after shifting in x) the two functions h_1 and h_2 coincide with the solution h constructed in Proposition 7.5 having a unique global maximum at $x = 0$. This implies that condition (A2) fixes the shift in x and thus guarantees uniqueness.

It remains to show that (8.1) holds. For this, we introduce the auxiliary function

$$\Psi := \varphi\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} - \frac{1}{2}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right)^2. \tag{8.2}$$

We need to show that $\Psi \geqslant 0$ for $x \in [x_c, \infty)$. We first calculate

$$\frac{d\Psi}{dx} = \varphi \frac{d^3\varphi}{dx^3} \overset{(4.2)}{=} (h_1 - h_2)\left(F(h_1) - F(h_2)\right) \geqslant 0,$$

since $F(h)$ is strictly increasing in h, see (4.5). It is hence sufficient to show that $\Psi(x) \to 0$ as $x \searrow x_c$. By assumption, $\varphi \in C^1([x_c, \infty))$ and $\varphi(x_c) = \frac{d\varphi}{dx}(x_c) = 0$ and hence $\left(\frac{d\varphi}{dx}(x)\right)^2 \to 0$ as $x \to x_c$. We will show that $(\varphi \frac{d^2\varphi}{dx^2})(x_k) \to 0$ for some sequence $x_k \searrow x_c$. This yields $\Psi(x) \searrow 0$ as $x \searrow x_c$ and thus proves our claim because Ψ is monotonically increasing in x.

We fix $i \in \{1, 2\}$. For $\delta > 0$ sufficiently small, we have $h_i < A$ and hence $F(h_i) < 0$ in $I_\delta := (x_c, x_c + \delta)$. It follows that $\frac{d^2 h_i}{dx^2}$ is monotonically decreasing in I_δ (in particular oscillations are excluded) and because of $h(x_c) = \frac{dh_i}{dx}(x_c) = 0$ we necessarily have

$$h_i(x) = o\left(x - x_c\right), \quad \frac{dh_i}{dx}(x) = o(1), \quad \text{and} \quad \frac{d^2 h_i}{dx^2}(x) = o\left((x - x_c)^{-1}\right) \quad \text{as } x \searrow x_c.$$

Because of $\varphi = h_1 - h_2$, this amounts to

$$\Psi \overset{(8.2)}{=} \varphi \frac{d^2\varphi}{dx^2} - \frac{1}{2}\left(\frac{d\varphi}{dx}\right)^2 \to 0 \qquad \text{as } x \searrow x_c.$$

□

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Appendix. Governing equations of the spin coating problem

In this section we derive the evolution equation (1.1). The derivation is mainly standard. Similar discussions can be found for instance in [8, 10, 25]. For axisymmetric solutions, we use the lubrication approximation regime which allows us to ignore gravitation and Coriolis force. But, at the beginning of the derivation, we include both forces to enhance understanding. We start from the incompressible Navier-Stokes equations describing the fluid flow in the rotating frame of reference as

$$\begin{aligned} \rho\left(\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u}\right) &= -\nabla p + \mu \Delta \mathbf{u} - \rho \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{u}) - 2\rho \boldsymbol{\omega} \times \mathbf{u}, \\ \nabla \cdot \mathbf{u} &= 0, \end{aligned} \tag{1}$$

where $\mathbf{u}$ is the fluid velocity consisting of u_r, u_θ, u_z as velocity components in each direction of the cylindrical coordinate (r, θ, z) and $\boldsymbol{\omega} = (0, 0, \omega)$ is the angular velocity of the rotation. On the right hand side, the third and fourth terms describe the centrifugal force and the Coriolis force, represented by

$$-\rho\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{u}) = -\nabla\Big(\frac{\rho\omega^2r^2}{2}\Big), \qquad -2\rho\boldsymbol{\omega}\times\mathbf{u} = -2\rho\omega(-u_\theta\mathbf{r}+u_r\boldsymbol{\theta}),$$

respectively ($\mathbf{r},\boldsymbol{\theta}$ are unit vectors along r,θ directions). In components, (1) turns into

$$\rho\Big(\partial_t u_r + u_r(\partial_r u_r) + \frac{u_\theta}{r}(\partial_\theta u_r) + u_z(\partial_z u_r) - \frac{u_\theta^2}{r}\Big)$$
$$= -\partial_r p_R + \mu\Big(\Delta u_r - \frac{u_r}{r^2} - \frac{2}{r^2}(\partial_\theta u_\theta)\Big) + 2\rho\omega u_\theta,$$

$$\rho\Big(\partial_t u_\theta + u_r(\partial_r u_\theta) + \frac{u_\theta}{r}(\partial_\theta u_\theta) + u_z(\partial_z u_\theta) - \frac{u_\theta u_r}{r}\Big)$$
$$= -\frac{1}{r}(\partial_\theta p_R) + \mu\Big(\Delta u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2}(\partial_\theta u_r)\Big) - 2\rho\omega u_r,$$

$$\rho\left(\partial_t u_z + u_r(\partial_r u_z) + \frac{u_\theta}{r}(\partial_\theta u_z) + u_z(\partial_z u_z)\right) = -\partial_z p_R + \mu\Delta u_z,$$

where $p_R = p + \rho g z - \frac{\rho}{2}\omega^2 r^2$ and where the continuity equation takes the form

$$\frac{1}{r}\partial_r(r u_r) + \frac{1}{r}(\partial_\theta u_\theta) + \partial_z u_z = 0. \tag{2}$$

We assume the Navier-slip boundary condition and the condition for no flux through the solid boundary, i.e., $(u_r,\, u_\theta)_{|z=0} = \frac{\lambda}{3}\,(\partial_z u_r,\, \partial_z u_\theta)_{|z=0}$ and $u_{z|z=0} = 0$, where $\frac{\lambda}{3}$ is the slip length. For the surface function $z = h(r,\theta,t)$ of the fluid interface, the kinematic boundary condition is given by $u_z = \partial_t h + u_r(\partial_r h) + \frac{u_\theta}{r}(\partial_\theta h)$, and normal stress is represented by $\left[-pI + \mu(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)\right]\mathbf{n} = \kappa\sigma\mathbf{n} + \nabla_S\sigma$, where $\kappa = -\nabla\cdot\mathbf{n}$ is the mean curvature of the interface, σ is the surface tension and $\mathbf{n}$ is the unit normal vector.

Let H, R be characteristic thickness and radius of the film and assume $\delta := \frac{H}{R} \ll 1$. Denote by U_r, U_θ, U_z the velocity components in the r,θ,z-directions, respectively. We also introduce the dimensionless variables $\bar r = \frac{r}{R}$, $\bar z = \frac{z}{H}$, $\bar t = \frac{U_r t}{R}$, $\bar\omega = \frac{\omega}{\Omega}$, and $\bar u_\alpha = \frac{u_\alpha}{U_\alpha}$ for $\alpha \in \{r,\theta,z\}$. The continuity equation (2) is re-scaled as $\frac{1}{\bar r}\partial_{\bar r}(\bar r\bar u_r) + \frac{U_\theta}{U_r}\frac{1}{\bar r}(\partial_{\bar\theta}\bar u_\theta) + \frac{U_z}{\delta U_r}(\partial_{\bar z}\bar u_z) = 0$, implying $U_z = \delta U_r$. We choose $U_\theta = \frac{\rho\Omega H^2}{\mu}U_r$ in order to keep the Coriolis force in the angular direction. Furthermore, we represent the scaled pressure $p_R = \frac{\mu U_r R}{H^2}\bar p_R$ with $\bar p_R = \frac{H^2}{\mu U_r R}(p + \rho g H\bar z - \frac{\rho}{2}\Omega^2R^2\bar\omega^2\bar r^2) = \bar p + \mathrm{St}^{-1}\bar z - \frac{\bar\omega^2\bar r^2}{2}$, where $\mathrm{St} = \frac{\mu U_r}{\delta\rho g H^2}$ is the Stokes number. It follows $U_r = \frac{\rho\Omega^2 R H^2}{\mu}$ to keep the effect of the centrifugal force.

Now let us consider the axisymmetric problem. We consider the regime when both the Reynolds number $\mathrm{Re} = \frac{\rho U_r R}{\mu} \sim O(1)$ and the Capillary number $\mathrm{Ca} = \delta^{-3}\frac{\mu U_r}{\sigma} \sim O(1)$ are $O(1)$; this is the so called lubrication approximation regime. Furthermore, when we consider dominant viscosity with respect to the gravity, the Navier-Stokes equations in leading order are given by

$$\partial_z^2 u_r = \partial_r p - \omega^2 r, \qquad \partial_z^2 u_\theta = 2\omega u_r, \qquad \partial_z p = 0, \tag{3}$$

the continuity equation is $\frac{1}{r}\partial_r(ru_r)+\partial_z u_z = 0$ (we have skipped the bars in the notation). The boundary conditions at $z = 0$ are $(u_r, u_\theta) = \frac{\lambda}{3H}(\partial_z u_r, \partial_z u_\theta)$, $u_z = 0$, the kinematic boundary condition at the free surface $z = h(r, t)$ is $u_z = \partial_t h + u_r(\partial_r h)$. We rewrite the stress conditions at $z = h(r, t)$ as

$$p = -\frac{\mathrm{Ca}^{-1}}{r}\partial_r(r(\partial_r h)), \qquad \partial_z u_r = 0, \qquad \partial_z u_\theta = 0. \tag{4}$$

By means of the continuity equation and boundary conditions, we find the relation $\partial_t h + \frac{1}{r}\,\partial_r\big(\int_0^h ru_r dz\big) = 0$. Together with (3) and (4) we then obtain the nondimensional thin-film equation in highest order

$$\partial_t h + r^{-1}\partial_r\Big(\omega^2 r^2\Big(\frac{h^3}{3} + \frac{\lambda h^2}{3H}\Big)\Big) + r^{-1}\partial_r\Big(\mathrm{Ca}^{-1} r\Big(\frac{h^3}{3} + \frac{\lambda h^2}{3H}\Big)\partial_r\Big(r^{-1}\partial_r(r(\partial_r h))\Big)\Big).$$

Written in the original variables the thin-film equation is given by

$$\partial_t h + \left(\frac{\rho\omega^2}{3\mu}\right) r^{-1}\partial_r\left(r^2 m(h)h\right) + \left(\frac{\sigma}{3\mu}\right) r^{-1}\partial_r\left(rm(h)h\partial_r\left(r^{-1}\partial_r(r(\partial_r h))\right)\right) = 0, \tag{5}$$

where $m(h) = h^2 + \lambda h$ is the mobility. With variables $\epsilon = \frac{\sigma}{\rho\omega^2}$, and $t \mapsto \frac{3\mu}{\rho\omega^2}t$, the thin-film equation (5) can be transformed to (1.1).

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