Corrigendum to "Smooth zero-contact-angle solutions to a thin-film equation around the steady state" [Journal of Differential Equations 245 (2008), 1454-1506]

Björn Bringmann[a,*], Lorenzo Giacomelli[b], Hans Knüpfer[c], Felix Otto[d]

[a] *Fakultät für Mathematik, Technische Universität München, Boltzmannstr. 3, 85748 Garching b. München, Germany*

[b] *SBAI Department, Sapienza University of Rome, Via Scarpa 16, 00161 Roma, Italy*

[c] *Institute of Applied Mathematics, University of Heidelberg, Im Neuenheimer Feld 294, 69120 Heidelberg, Germany*

[d] *Max Planck Institute for Mathematics in the Sciences, Inselstr. 22, 04103 Leipzig, Germany*

Abstract

This paper is a corrigendum to "Smooth zero-contact-angle solutions to a thin-film equation around the steady state" [Journal of Differential Equations 245 (2008), 1454-1506].

Keywords: Existence and uniqueness, Fourth order degenerate parabolic equations, Thin-film equations, Free boundary problems, Thin fluid films, Lubrication theory, Hele-Shaw flow

2000 MSC: 35A05, 35K65, 35Q35, 35R35, 76A20, 76D08, 76D27

The aim of this note is to close a gap in the proofs of Lemma 6.1 and Lemma 6.2 in [?]. More precisely, in the proof of estimate (92) in [?], we have used the assumption that $\partial_x^3 G_{|x=0} = 0$. This can not be done in general, since the function G in (92) in fact represents the second term, G_+, of the decomposition $G = G_- + G_+$ appearing in the definition of interpolation norms, see e.g. (3) below. An analogous gap was present in the proof of Lemma 6.2. We also improve the definition of the function spaces X_m^* (resp. X_m) to which solutions of the nonlinear (resp. linear) problems belong and discuss some technical issue related to them.

Lemmata 6.1 and 6.2

We begin by providing a complete proof of the original lemmata. Then, we sketch how a slightly weaker statement can be concluded from tools already present in [?], which is nevertheless strong enough to deduce the theorems in [?]. In fact, the complete proof leads to a slightly more general formulation of Lemma 6.1, which does not require $\partial_x^3 G|_{x=0} = 0$:

*Corresponding author

Email addresses: bjoern.bringmann@tum.de (Björn Bringmann), lorenzo.giacomelli@sbai.uniroma1.it (Lorenzo Giacomelli), knuepfer@uni-heidelberg.de (Hans Knüpfer), otto@mis.mpg.de (Felix Otto)

Lemma 6.1. *For any given $F, G \in C_c^\infty([0,\infty)^2)$ with $F_{|x=0} = 0$, we have*

$$\left[\partial_x \left((F - x\partial_x F_{|x=0})\partial_x^3 G\right)\right]_{L^2(H_2)^*} \lesssim [F]_{C^0(H_4)^*}\, [G]_{L^2(H_6)^*}\,. \tag{1}$$

This slightly more general formulation in turn permits to simplify the statement and the proof of Lemma 6.2 (where an analogous gap was present):

Lemma 6.2. *For any given $F, G \in C_c^\infty([0,\infty)^2)$ with $F_{|x=0} = 0$, we have*

$$\left[\partial_x \left(F - x\partial_x F_{|x=0}\right) \partial_x^3 G|_{x=0}\right]_{L^2(H_2)^*} \lesssim [F]_{C^0(H_4)^*}\, [G]_{L^2(H_6)^*}\,. \tag{2}$$

In [**?**], Lemma 6.1 and 6.2 are only used to prove Proposition 2.2, whose proof is now an immediate consequence of these new formulations.

Preliminaries. We use the same notations as in [**?**]. In particular, we write for the spatial semi-norms

$$[F]_{H_m} := \left(\int_0^\infty x^{m-1} (\partial_x^m F)^2 dx\right)^{\frac{1}{2}},$$

for the space-time semi-norms

$$[F]_{C^0(H_m)} := \sup_{t \in (0,\infty)} [F(t)]_{H_m} \quad \text{and} \quad [F]_{L^2(H_m)} := \left(\int_0^\infty [F(t)]_{H_m}^2 dt\right)^{\frac{1}{2}},$$

and for the space-time interpolation norms

$$[F]_{L^2(H_m)^*} := \int_0^\infty \inf_{F = F_-(s) + F_+(s)} \left(s^{-1}[F_-(s)]_{L^2(H_{m-1})}^2 + s[F_+(s)]_{L^2(H_{m+1})}^2\right)^{\frac{1}{2}} \frac{ds}{s}, \tag{3}$$

$$[F]_{C^0(H_m)^*} := \int_0^\infty \inf_{F = F_-(s) + F_+(s)} \left(s^{-1}[F_-(s)]_{C^0(H_{m-1})}^2 + s[F_+(s)]_{C^0(H_{m+1})}^2\right)^{\frac{1}{2}} \frac{ds}{s}. \tag{4}$$

Since $(a+b)/\sqrt{2} \le (a^2+b^2)^{1/2} \le a+b$ for all $a, b \ge 0$, by a change of the integration variable one sees immediately that

$$[F]_{L^2(H_m)^*} \sim \int_0^\infty \inf_{F = F_-(s) + F_+(s)} \left(s^{-1}[F_-(s)]_{L^2(H_{m-1})} + s[F_+(s)]_{L^2(H_{m+1})}\right) \frac{ds}{s}, \tag{5}$$

$$[F]_{C^0(H_m)^*} \sim \int_0^\infty \inf_{F = F_-(s) + F_+(s)} \left(s^{-1}[F_-(s)]_{C^0(H_{m-1})} + s[F_+(s)]_{C^0(H_{m+1})}\right) \frac{ds}{s}. \tag{6}$$

We will appeal to the fact that there are two equivalent (rather, dual) ways to define an interpolation norm (see for instance [**?** , Theorem 3.3.1]): Next to the definition via the K-functional we have been working with, namely (5)-(6), there is also the definition via the J-functional:

$$[F]_{L^2(H_m)^*} \sim \inf\Big\{ \int_0^\infty \max\{s^{-1}[F(s)]_{L^2(H_{m-1})}, s[F(s)]_{L^2(H_{m+1})}\} \frac{ds}{s} \;:\; F = \int_0^\infty F(s)\, \frac{ds}{s}\Big\}, \tag{7}$$

$$[F]_{C^0(H_m)^*} \sim \inf\Big\{ \int_0^\infty \max\{s^{-1}[F(s)]_{C^0(H_{m-1})}, s[F(s)]_{C^0(H_{m+1})}\} \frac{ds}{s} \;:\; F = \int_0^\infty F(s)\, \frac{ds}{s}\Big\}. \tag{8}$$

Finally, we note the following immediate consequence of (3):

$$[F]_{L^2(H_2)^*} \;\le\; \int_0^\infty \min\left\{s^{-1}[F]_{L^2(H_1)}, s[F]_{L^2(H_3)}\right\} \frac{ds}{s}. \tag{9}$$

The corresponding seminorms on a time interval are denoted by $[\cdot]_{L^2(I,H_m)^*}$, $[\cdot]_{C^0(I,H_m)^*}$. The interpolation norms and spaces are defined exactly as in the case of half space.

Proof of Lemma 6.1. Incorporating the boundary conditions $F_{|x=0} = 0$, we may redefine the function on the left-hand side of (1) by

$$\mathcal{N}_1(F,G) = \partial_x((F - F_{|x=0} - x\partial_x F_{|x=0})\partial_x^3 G). \tag{10}$$

Because of (7) and (8), we may disintegrate F and G into families $\{F(r)\}_{r>0}$ and $\{G(s)\}_{s>0}$, respectively, i. e.

$$F = \int_0^\infty F(r)\;\frac{dr}{r} \quad \text{and} \quad G = \int_0^\infty G(s)\;\frac{ds}{s}, \tag{11}$$

such that

$$\int_0^\infty \max\left\{r^{-1}[F(r)]_{C^0(H_3)}, r[F(r)]_{C^0(H_5)}\right\} \frac{dr}{r} \;\lesssim\; [F]_{C^0(H_4)^*}, \tag{12}$$

$$\int_0^\infty \max\left\{s^{-1}[G(s)]_{L^2(H_5)}, s[G(s)]_{L^2(H_7)}\right\} \frac{ds}{s} \;\lesssim\; [G]_{L^2(H_6)^*}. \tag{13}$$

Because of (11) and the bilinearity of $\mathcal{N}_1(\cdot,\cdot)$ in conjunction with the triangle inequality for $[\cdot]_{L^2(H_2)^*}$, we have

$$\begin{aligned}
[\mathcal{N}_1(F,G)]_{L^2(H_2)^*} &= \int_0^\infty\int_0^\infty [\mathcal{N}_1(F(r),G(s))]_{L^2(H_2)^*}\;\frac{dr}{r}\frac{ds}{s} \qquad (14)\\
&\overset{(9)}{\le} \int_0^\infty\int_0^\infty\int_0^\infty \min\{\sigma^{-1}[\mathcal{N}_1(F(r),G(s))]_{L^2(H_1)}, \sigma[\mathcal{N}_1(F(r),G(s))]_{L^2(H_3)}\}\;\frac{d\sigma}{\sigma}\frac{dr}{r}\frac{ds}{s}.
\end{aligned}$$

We divide the proof into two steps.

Step 1: We first claim that the two estimates

$$\begin{aligned}
[\mathcal{N}_1(F,G)]_{L^2(H_1)} &\lesssim \min\Big\{\left([F]_{C^0(H_3)}[F]_{C^0(H_5)}\right)^{\frac12}[G]_{L^2(H_5)},\\
&\qquad [F]_{C^0(H_3)}\left([G]_{L^2(H_5)}[G]_{L^2(H_7)}\right)^{\frac12}\Big\}, \qquad (15)\\
[\mathcal{N}_1(F,G)]_{L^2(H_3)} &\lesssim \max\Big\{\left([F]_{C^0(H_3)}[F]_{C^0(H_5)}\right)^{\frac12}[G]_{L^2(H_7)},\\
&\qquad [F]_{C^0(H_5)}\left([G]_{L^2(H_5)}[G]_{L^2(H_7)}\right)^{\frac12}\Big\}, \qquad (16)
\end{aligned}$$

when applied to $F(r)$ and $G(s)$, are enough to connect (14) to (12) and thus prove (1).

Indeed, we first insert (15)–(16) into the integrand of (14) to obtain

$$\begin{aligned}
&\min\Big\{\sigma^{-1}[\mathcal{N}_1(F(r),G(s))]_{L^2(H_1)},\sigma[\mathcal{N}_1(F(r),G(s))]_{L^2(H_3)}\Big\}\\
&\quad\lesssim \min\Big\{\sigma^{-1}\min\Big\{\,\big([F(r)]_{C^0(H_3)}[F(r)]_{C^0(H_5)}\big)^{\frac12}[G(s)]_{L^2(H_5)},\\
&\qquad\qquad [F(r)]_{C^0(H_3)}\big([G(s)]_{L^2(H_5)}[G(s)]_{L^2(H_7)}\big)^{\frac12}\,\Big\},\\
&\qquad\sigma\max\Big\{\,\big([F(r)]_{C^0(H_3)}[F(r)]_{C^0(H_5)}\big)^{\frac12}[G(s)]_{L^2(H_7)},\\
&\qquad\qquad [F(r)]_{C^0(H_5)}\big([G(s)]_{L^2(H_5)}[G(s)]_{L^2(H_7)}\big)^{\frac12}\,\Big\}\Big\}
\end{aligned}$$

for any $r,s,\sigma>0$. A straightforward calculation shows that for any $A,B,C,D\in\mathbb{R}$, we have $\min\{\min\{A,B\},\max\{C,D\}\}\le\max\{\min\{A,C\},\min\{B,D\}\}$. Applying this inequality in form of $\min\{\min\{A,B\},\max\{C,D\}\}\le\min\{A,C\}+\min\{B,D\}$, we thus obtain

$$\begin{aligned}
&\min\Big\{\sigma^{-1}[\mathcal{N}_1(F(r),G(s))]_{L^2(H_1)},\sigma[\mathcal{N}_1(F(r),G(s))]_{L^2(H_3)}\Big\}\\
&\quad\lesssim \big([F(r)]_{C^0(H_3)}[F(r)]_{C^0(H_5)}\big)^{\frac12}\min\Big\{\sigma^{-1}[G(s)]_{L^2(H_5)},\sigma[G(s)]_{L^2(H_7)}\Big\}\\
&\qquad+\min\Big\{\sigma^{-1}[F(r)]_{C^0(H_3)},\sigma[F(r)]_{C^0(H_5)}\Big\}\big([G(s)]_{L^2(H_5)}[G(s)]_{L^2(H_7)}\big)^{\frac12}\\
&\quad\le\max\Big\{r^{-1}[F(r)]_{C^0(H_3)},r[F(r)]_{C^0(H_5)}\Big\}\min\Big\{\sigma^{-1}[G(s)]_{L^2(H_5)},\sigma[G(s)]_{L^2(H_7)}\Big\}\\
&\qquad+\min\Big\{\sigma^{-1}[F(r)]_{C^0(H_3)},\sigma[F(r)]_{C^0(H_5)}\Big\}\max\Big\{s^{-1}[G(s)]_{L^2(H_5)},s[G(s)]_{L^2(H_7)}\Big\}
\end{aligned}\tag{17}$$

for any $r,s,\sigma>0$ where we used Young's inequality in the last step. Integrating (17) over σ, r, and s, we obtain

$$\begin{aligned}
&\int_0^\infty\int_0^\infty\int_0^\infty\min\left\{\sigma^{-1}[\mathcal{N}_1(F(r),G(s))]_{L^2(H_1)},\sigma[\mathcal{N}_1(F(r),G(s))]_{L^2(H_3)}\right\}\frac{d\sigma}{\sigma}\frac{dr}{r}\frac{ds}{s}\\
&\quad\lesssim\int_0^\infty\max\left\{r^{-1}[F(r)]_{C^0(H_3)},r[F(r)]_{C^0(H_5)}\right\}\frac{dr}{r}\\
&\qquad\cdot\int_0^\infty\int_0^\infty\min\left\{\sigma^{-1}[G(s)]_{L^2(H_5)},\sigma[G(s)]_{L^2(H_7)}\right\}\frac{d\sigma}{\sigma}\frac{ds}{s}\\
&\quad+\int_0^\infty\int_0^\infty\min\left\{\sigma^{-1}[F(r)]_{C^0(H_3)},\sigma[F(r)]_{C^0(H_5)}\right\}\frac{d\sigma}{\sigma}\frac{dr}{r}\\
&\qquad\cdot\int_0^\infty\max\left\{s^{-1}[G(s)]_{L^2(H_5)},s[G(s)]_{L^2(H_7)}\right\}\frac{ds}{s}.
\end{aligned}\tag{18}$$

In order to recover the product of (12) and (13) on the right hand side of (18), it suffices to note (by a simple computation) that for all $A,B\ge0$ and $s>0$, we have

$$\int_0^\infty\min\left\{\sigma^{-1}A,\sigma B\right\}\frac{d\sigma}{\sigma}=2(AB)^{1/2}\le2\max\{s^{-1}A,sB\}.$$

Step 2: It remains to establish estimates (15) and (16), which we rewrite as

$$\begin{aligned}
&[\mathcal{N}_1(F,G)]_{L^2(H_1)}\lesssim\big([F]_{C^0(H_3)}[F]_{C^0(H_5)}\big)^{\frac12}[G]_{L^2(H_5)},\\
&[\mathcal{N}_1(F,G)]_{L^2(H_1)}\lesssim[F]_{C^0(H_3)}\big([G]_{L^2(H_5)}[G]_{L^2(H_7)}\big)^{\frac12},\\
&[\mathcal{N}_1(F,G)]_{L^2(H_3)}\lesssim\big([F]_{C^0(H_3)}[F]_{C^0(H_5)}\big)^{\frac12}[G]_{L^2(H_7)}+[F]_{C^0(H_5)}\big([G]_{L^2(H_5)}[G]_{L^2(H_7)}\big)^{\frac12}.
\end{aligned}$$

Written in this form, we see from Cauchy-Schwarz' inequality in the time variable that it is enough to establish the *spatial* estimates

$$[\mathcal{N}_1(F,G)]_{H_1} \lesssim ([F]_{H_3}[F]_{H_5})^{\frac12}\,[G]_{H_5}, \tag{19}$$

$$[\mathcal{N}_1(F,G)]_{H_1} \lesssim [F]_{H_3}\,([G]_{H_5}[G]_{H_7})^{\frac12}, \tag{20}$$

$$[\mathcal{N}_1(F,G)]_{H_3} \lesssim ([F]_{H_3}[F]_{H_5})^{\frac12}\,[G]_{H_7} + [F]_{H_5}\,([G]_{H_5}[G]_{H_7})^{\frac12}. \tag{21}$$

We start with (19)–(20) and obtain by Leibniz' rule

$$\begin{aligned}[\mathcal{N}_1(F,G)]_{H_1} &\overset{(10)}{=} \|\partial_x^2((F - F_{|x=0} - x\partial_x F_{|x=0})\partial_x^3 G)\|_{L^2}\\ &\lesssim \|\partial_x^2 F\partial_x^3 G\|_{L^2} + \|(\partial_x F - \partial_x F_{|x=0})\partial_x^4 G\|_{L^2} + \|(F - F_{|x=0} - x\partial_x F_{|x=0})\partial_x^5 G\|_{L^2}.\end{aligned}$$

In order to obtain the estimate (19) against $([F]_{H_3}[F]_{H_5})^{1/2}[G]_{H_5}$, we use

$$\begin{aligned}[\mathcal{N}_1(F,G)]_{H_1} \lesssim\ & \|\partial_x^2 F\|_{L^\infty}\|\partial_x^3 G\|_{L^2} + \|x^{-1}(\partial_x F - \partial_x F_{|x=0})\|_{L^\infty}\|x\partial_x^4 G\|_{L^2}\\ &+ \|x^{-2}(F - F_{|x=0} - x\partial_x F_{|x=0})\|_{L^\infty}\|x^2\partial_x^5 G\|_{L^2},\end{aligned} \tag{22}$$

while in order to estimate (20) against $[F]_{H_3}([G]_{H_5}[G]_{H_7})^{1/2}$, we use

$$\begin{aligned}[\mathcal{N}_1(F,G)]_{H_1} \lesssim\ & \|\partial_x^2 F\|_{L^2}\|\partial_x^3 G\|_{L^\infty} + \|x^{-\frac12}(\partial_x F - \partial_x F_{|x=0})\|_{L^\infty}\|x^{\frac12}\partial_x^4 G\|_{L^2}\\ &+ \|x^{-\frac32}(F - F_{|x=0} - x\partial_x F_{|x=0})\|_{L^\infty}\|x^{\frac32}\partial_x^5 G\|_{L^2}.\end{aligned} \tag{23}$$

Using Taylor's theorem on the F factor, (22) can be post-processed to

$$[\mathcal{N}_1(F,G)]_{H_1} \lesssim \|\partial_x^2 F\|_{L^\infty}\big(\|\partial_x^3 G\|_{L^2} + \|x\partial_x^4 G\|_{L^2} + \|x^2\partial_x^5 G\|_{L^2}\big),$$

while using the fundamental theorem of calculus and Hölder inequality, (23) can be post-processed to

$$[\mathcal{N}_1(F,G)]_{H_1} \lesssim \|\partial_x^2 F\|_{L^2}\big(\|\partial_x^3 G\|_{L^\infty} + \|x^{\frac12}\partial_x^4 G\|_{L^2} + \|x^{\frac32}\partial_x^5 G\|_{L^2}).$$

The $\|\cdot\|_{L^\infty}$-norms can be estimated by $\|\cdot\|_{L^2}$-norms by the standard estimate:

$$\begin{aligned}[\mathcal{N}_1(F,G)]_{H_1} &\lesssim \big(\|\partial_x^2 F\|_{L^2}\|\partial_x^3 F\|_{L^2}\big)^{\frac12}\big(\|\partial_x^3 G\|_{L^2} + \|x\partial_x^4 G\|_{L^2} + \|x^2\partial_x^5 G\|_{L^2}\big),\\ [\mathcal{N}_1(F,G)]_{H_1} &\lesssim \|\partial_x^2 F\|_{L^2}\big(\big(\|\partial_x^3 G\|_{L^2}\|\partial_x^4 G\|_{L^2}\big)^{\frac12} + \|x^{\frac12}\partial_x^4 G\|_{L^2} + \|x^{\frac32}\partial_x^5 G\|_{L^2}).\end{aligned}$$

In the second estimate, we get rid of the fractional powers in the weight with help of Cauchy-Schwarz:

$$\begin{aligned}[\mathcal{N}_1(F,G)]_{H_1} \lesssim\ & \|\partial_x^2 F\|_{L^2}\big(\big(\|\partial_x^3 G\|_{L^2}\|\partial_x^4 G\|_{L^2}\big)^{\frac12} + \big(\|x\partial_x^4 G\|_{L^2}\|\partial_x^4 G\|_{L^2}\big)^{\frac12}\\ &+ \big(\|x^2\partial_x^5 G\|_{L^2}\|x\partial_x^5 G\|_{L^2}\big)^{\frac12}\big).\end{aligned}$$

Using Hardy's estimate, this yields (19)–(20), i.e.

$$\begin{aligned}[\mathcal{N}_1(F,G)]_{H_1} &\lesssim \big(\|x\partial_x^3 F\|_{L^2}\|x^2\partial_x^5 F\|_{L^2}\big)^{\frac12}\,\|x^2\partial_x^5 G\|_{L^2} = ([F]_{H_3}[F]_{H_5})^{\frac12}\,[G]_{H_5},\\ [\mathcal{N}_1(F,G)]_{H_1} &\lesssim \|x\partial_x^3 F\|_{L^2}\,\big(\|x^2\partial_x^5 G\|_{L^2}\|x^3\partial_x^7 G\|_{L^2}\big)^{\frac12} = [F]_{H_3}\,([G]_{H_5}[G]_{H_7})^{\frac12}.\end{aligned}$$

We finally address (21). By Leibniz' rule we have

$$\begin{aligned}[\mathcal{N}_1(F,G)]_3 &\overset{(10)}{=} \|x\partial_x^4((F-F_{|x=0}-x\partial_xF_{|x=0})\partial_x^3G)\|_{L^2}\\ &\lesssim \|x\partial_x^4F\partial_x^3G\|_{L^2}+\|x\partial_x^3F\partial_x^4G\|_{L^2}+\|x\partial_x^2F\partial_x^5G\|_{L^2}\\ &\qquad+\|x(\partial_xF-\partial_xF_{|x=0})\partial_x^6G\|_{L^2}+\|x(F-F_{|x=0}-x\partial_xF_{|x=0})\partial_x^7G\|_{L^2}\end{aligned}$$

so that (21) follows from the two estimates

$$\|x\partial_x^4F\partial_x^3G\|_{L^2}+\|x\partial_x^3F\partial_x^4G\|_{L^2}\lesssim [F]_{H_5}\left([G]_{H_5}[G]_{H_7}\right)^{\frac12}, \tag{24}$$

$$\begin{aligned}&\|x\partial_x^2F\partial_x^5G\|_{L^2}+\|x(\partial_xF-\partial_xF_{|x=0})\partial_x^6G\|_{L^2}\\ &\quad+\|x(F-F_{|x=0}-x\partial_xF_{|x=0})\partial_x^7G\|_{L^2}\lesssim \left([F]_{H_3}[F]_{H_5}\right)^{\frac12}[G]_{H_7}.\end{aligned} \tag{25}$$

Here is the argument for (24): Using the same tools as for (15) —- using the estimates $\|fg\|_{L^2}\le\|f\|_{L^2}\|g\|_{L^\infty}$, $\|g\|_{L^\infty}\le(\|g\|_{L^2}\|\partial_xg\|_{L^2})^{1/2}$, and Hardy's inequality — we obtain as desired

$$\begin{aligned}&\|x\partial_x^4F\partial_x^3G\|_{L^2}+\|x\partial_x^3F\partial_x^4G\|_{L^2}\\ &\quad\lesssim \|x\partial_x^4F\|_{L^2}\|\partial_x^3G\|_{L^\infty}+\|\partial_x^3F\|_{L^2}\|x\partial_x^4G\|_{L^\infty}\\ &\quad\lesssim \|x\partial_x^4F\|_{L^2}\left(\|\partial_x^3G\|_{L^2}\|\partial_x^4G\|_{L^2}\right)^{\frac12}+\|\partial_x^3F\|_{L^2}\left(\|x\partial_x^4G\|_{L^2}(\|x\partial_x^5G\|_{L^2}+\|\partial_x^4G\|_{L^2})\right)^{\frac12}\\ &\quad\lesssim \|x^2\partial_x^5F\|_{L^2}\left(\|x^2\partial_x^5G\|_{L^2}\|x^3\partial_x^7G\|_{L^2}\right)^{\frac12}.\end{aligned}$$

The argument for (25) follows the same lines and uses Taylor's theorem on the F factor:

$$\begin{aligned}&\|x\partial_x^2F\partial_x^5G\|_{L^2}+\|x(\partial_xF-\partial_xF_{|x=0})\partial_x^6G\|_{L^2}+\|x(F-F_{|x=0}-x\partial_xF_{|x=0})\partial_x^7G\|_{L^2}\\ &\quad\le \|\partial_x^2F\|_{L^\infty}\|x\partial_x^5G\|_{L^2}+\|x^{-1}(\partial_xF-\partial_xF_{|x=0})\|_{L^\infty}\|x^2\partial_x^6G\|_{L^2}\\ &\qquad\quad+\|x^{-2}(F-F_{|x=0}-x\partial_xF_{|x=0})\|_{L^\infty}\|x^3\partial_x^7G\|_{L^2}\\ &\quad\lesssim \|\partial_x^2F\|_{L^\infty}(\|x\partial_x^5G\|_{L^2}+\|x^2\partial_x^6G\|_{L^2}+\|x^3\partial_x^7G\|_{L^2})\\ &\quad\lesssim \left(\|\partial_x^2F\|_{L^2}\|\partial_x^3F\|_{L^2}\right)^{\frac12}(\|x\partial_x^5G\|_{L^2}+\|x^2\partial_x^6G\|_{L^2}+\|x^3\partial_x^7G\|_{L^2})\\ &\quad\lesssim \left(\|x\partial_x^3F\|_{L^2}\|x^2\partial_x^5F\|_{L^2}\right)^{\frac12}\|x^3\partial_x^7G\|_{L^2}.\end{aligned}$$

□

Proof of Lemma 6.2. Let $\mathcal{N}_2(F,G)=\partial_x(F-x\partial_xF|_{x=0})\partial_x^3G|_{x=0}$. It follows from Lemma 1.2 and Lemma 1.3 in [**?**] that

$$\int_0^\infty \sup_{x\in(0,\infty)}|\partial_x^3G(t)|^2dt\lesssim [G]^2_{L^2(H_6)^*}. \tag{26}$$

By (3),

$$\begin{aligned}&[N_2(F,G)]_{L^2(H_2)^*}\\ &\quad\le\int_0^\infty \inf_{F=F_-(s)+F_+(s)}\left(s^{-1}[\mathcal{N}_2(F_-(s),G)]^2_{L^2(H_1)}+s[\mathcal{N}_2(F_+(s),G)]^2_{L^2(H_3)}\right)^{\frac12}\frac{ds}{s}.\end{aligned} \tag{27}$$

Using Hardy's inequality and (26), we have

$$[\mathcal{N}_2(F_-,G)]^2_{L^2(H_1)} = \int_0^\infty \int_0^\infty (\partial_x^2 F_-)^2(\partial_x^3 G|_{x=0})^2 dxdt \lesssim [F_-]^2_{C^0(H_3)}\,[G]^2_{L^2(H_6)^*}\,, \quad (28)$$

$$[\mathcal{N}_2(F_+,G)]^2_{L^2(H_3)} = \int_0^\infty \int_0^\infty x^2(\partial_x^4 F_+)^2(\partial_x^3 G|_{x=0})^2 dxdt \lesssim [F_+]^2_{C^0(H_5)}\,[G]^2_{L^2(H_6)^*} . \quad (29)$$

Inserting (28) and (29) into (27) we obtain (2). □

An alternative fix. We also sketch an alternative fix, which is based on the observation that in computing $[F]_{C^0(H_k)^*}$ and $[F]_{L^2(H_k)^*}$ the same interpolant can be used. Recall the definition of the linear operator A from [**?**], which is

$$AF := \frac{1}{2}\partial_x\left(x^2\partial_x^3 F\right) .$$

Lemma C.1. *Let $F \in C_c^\infty([0,\infty)^2)$ with $F|_{x=0} = 0$ and $T > 0$. For $s \geq 0$ define*

$$F_+(s;t,x) := \left[(\mathrm{I} + 2s^2 A)^{-1}F(t,\cdot)\right](x) \quad \textit{and} \quad F_-(s) = F - F_+(s) . \quad (30)$$

Then $F_\pm$ have the following properties:

1. *The interpolants $F_\pm(s)$ are regular, i.e.*

$$F_\pm(s) \in \bigcap_{k\in\mathbb{N}} L^2(H_k) \quad \textit{and} \quad F_\pm(s) \in \bigcap_{k\in\mathbb{N}} C^0(H_k) .$$

2. *For every $k \geq 2$ it holds that*

$$\int_0^\infty \left(s^{-1}[F_-(s)]^2_{L^2([0,T];H_{k-1})} + s[F_+(s)]^2_{L^2([0,T];H_{k+1})}\right)^{\frac{1}{2}} \frac{\mathrm{d}s}{s}$$
$$= [F]_{L^2([0,T];H_k)^*} .$$

3. *For every $k \geq 2$ it holds that*

$$\int_0^\infty \left(s^{-1}[F_-(s)]^2_{C^0([0,T];H_{k-1})} + s[F_+(s)]^2_{C^0([0,T];H_{k+1})}\right)^{\frac{1}{2}} \frac{\mathrm{d}s}{s}$$
$$\leq 2[F]_{C^0([0,T];H_k)^*} .$$

Proof. The regularity result can be deduced by using Proposition 4.1 in [**?**] together with the linearity of the operator A. By Lemma 3.2 in [**?**], the equality $2\langle AG_1, G_2\rangle_{H_{k-1}} = \langle G_1, G_2\rangle_{H_{k+1}}$ holds for all functions $G_1 \in H_{k+3}$ and $G_2 \in H_{k+1}$. Using this, we deduce for $\Psi \in L^2(H_{k+1})$ that

$$s^{-1}[F_-(s) - \Psi]^2_{L^2(H_{k-1})} + s[F_+(s) + \Psi]^2_{L^2(H_{k+1})}$$
$$= s^{-1}[F_-(s)]^2_{L^2(H_{k-1})} + s[F_+(s)]^2_{L^2(H_{k+1})} + s^{-1}[\Psi]^2_{L^2(H_{k-1})} + s[\Psi]^2_{L^2(H_{k+1})}$$
$$+ 2s^{-1}\langle -F + (\mathrm{Id} + 2s^2 A)F_+(s), \Psi\rangle_{L^2(H_{k-1})} .$$

By the definition of $F_+(s)$ the last summand is equal to zero. Thus we have

$$s^{-1}[F_-(s)]^2_{L^2(H_{k-1})} + s[F_+(s)]^2_{L^2(H_{k+1})} = \inf_{F=\Phi_-+\Phi_+} s^{-1}[\Phi_-(s)]^2_{L^2(H_{k-1})} + s[\Phi_+(s)]^2_{L^2(H_{k+1})} ,$$

which proves the second statement. Similar to the previous calculation, we can also deduce

$$s^{-1}[F_-(s;t)]^2_{H_{k-1}} + s[F_+(s;t)]^2_{H_{k+1}} = \inf_{F(t)=\Phi_-+\Phi_+} s^{-1}[\Phi_-]^2_{H_{k-1}} + s[\Phi_+]^2_{H_{k+1}} .$$

Given $\Phi_\pm \in C^0(H_{k\pm1})$ with $F = \Phi_- + \Phi_+$, it follows that

$$\begin{aligned} s^{-1}[F_-(s)]^2_{C^0(H_{k-1})} + s[F_+(s)]^2_{C^0(H_{k+1})} &\le 2\|s^{-1}[F_-(s)]^2_{H_{k-1}} + s[F_+(s)]^2_{H_{k+1}}\|_{C^0([0,\infty))} \\ \le 2\|s^{-1}[\Phi_-]^2_{H_{k-1}} + s[\Phi_+]^2_{H_{k+1}}\|_{C^0([0,\infty))} &\le 2\left(s^{-1}[\Phi_-]^2_{C^0(H_{k-1})} + s[\Phi_+]^2_{C^0(H_{k+1})}\right) . \end{aligned}$$

□

Proposition C.1. *For any given $F, G \in C_c^\infty([0,\infty)^2)$ with $F|_{x=0} = 0$, we have*

$$\begin{aligned} &[\partial_x \left((F - x\partial_x F|_{x=0})(\partial_x^3 G - \partial_x^3 G|_{x=0})\right)]_{L^2(H_2)^*} \\ &\quad \lesssim ([F]_{C^0(H_4)^*} + [F]_{L^2(H_6)^*})([G]_{C^0(H_4)^*} + [G]_{L^2(H_6)^*}) . \end{aligned}$$

Even though this estimate is slightly weaker than the original Proposition 2.2 in [**?**], the proofs of the main theorems remain essentially unchanged.

Proof. Let $F_\pm(s), G_\pm(s)$ be as in (30). Define

$$\Phi_\pm(s) := -\partial_x \left((F_\pm(s) - x\partial_x F_\pm(s)|_{x=0})\partial_x^3 G|_{x=0}\right) + \partial_x \left((F - x\partial_x F|_{x=0})\partial_x^3 G_\pm(s)\right) .$$

By using $\Phi_\pm$ as interpolants in computing $[N(F,G)]_{L^2(H_2)^*}$, in view of Lemma C.1 it suffices to show for $G_\pm = G_\pm(s)$ and $F_\pm = F_\pm(s)$ the following inequalities:

$$\left[\partial_x \left((F_- - x\partial_x F_-|_{x=0})\partial_x^3 G|_{x=0}\right)\right]_{L^2(H_1)} \lesssim [F_-]_{C^0(H_3)}[G]_{L^2(H_6^*)} , \tag{31}$$

$$\left[\partial_x \left((F_+ - x\partial_x F_+|_{x=0})\partial_x^3 G|_{x=0}\right)\right]_{L^2(H_3)} \lesssim [F_+]_{C^0(H_5)}[G]_{L^2(H_6^*)} , \tag{32}$$

$$\left[\partial_x \left((F - x\partial_x F|_{x=0})\partial_x^3 G_-\right)\right]_{L^2(H_1)} \lesssim [F]_{C^0(H_4^*)}[G_-]_{L^2(H_5)} , \tag{33}$$

$$\begin{aligned} \left[\partial_x \left((F - x\partial_x F|_{x=0})\partial_x^3 G_+\right)\right]_{L^2(H_3)} &\lesssim [F]_{C^0(H_4^*)}[G_+]_{L^2(H_7)} \\ &\quad + [F]_{L^2(H_6^*)}[G_+]_{C^0(H_5)} . \end{aligned} \tag{34}$$

The inequalities (31) and (32) have already been established in the new proof of Lemma 6.2. (33) can be handled as in the first part of the proof of [**?** , Lemma 6.1]. Let us now turn to the new estimate (34). As before, we prove a spatial version of the inequality. We estimate

$$\begin{aligned} &[\partial_x \left((F - x\partial_x F|_{x=0})\partial_x^3 G_+\right)]_{H_3} \\ &\quad = \|x\partial_x^4 \left((F - x\partial_x F|_{x=0})\partial_x^3 G_+\right)\|_{L^2} \\ &\quad \lesssim \|x\partial_x^4 F\partial_x^3 G_+\|_{L^2} + \|x\partial_x^3 F\partial_x^4 G_+\|_{L^2} + \|x\partial_x^2 F\partial_x^5 G_+\|_{L^2} \\ &\qquad + \|x(\partial_x F - \partial_x F|_{x=0})\partial_x^6 G_+\|_{L^2} + \|x(F - x\partial F|_{x=0})\partial_x^7 G_+\|_{L^2} \\ &\quad \le \|x\partial_x^4 F\partial_x^3 G_+\|_{L^2} + \|x\partial_x^3 F\|_{C^0}\|\partial_x^4 G_+\|_{L^2} + \|\partial_x^2 F\|_{C^0}\|x\partial_x^5 G_+\|_{L^2} \\ &\qquad + \|x^{-1}(\partial_x F - \partial_x F|_{x=0})\|_{C^0}\|x^2\partial_x^6 G_+\|_{L^2} + \|x^{-2}(F - x\partial_x F|_{x=0})\|_{C^0}\|x^3\partial_x^7 G_+\|_{L^2} . \end{aligned}$$

For the first summand we use the Hardy's inequality, Lemma 1.1 and Lemma A.2 in [?] to get

$$\begin{aligned}\|x\partial_x^4 F \partial_x^3 G_+\|_{L^2} &\le \|x^{\frac{1}{2}}\partial_x^3 G_+\|_{C^0}\|x^{\frac{1}{2}}\partial_x^4 F\|_{L^2} \lesssim \|x\partial_x^4 G_+\|_{L^2}\|x^{\frac{3}{2}}\partial_x^5 F\|_{L^2}\\ &\lesssim [F]_{H_6}[G_+]_{H_5} \lesssim [F]_{H_6^*}[G_+]_{H_5}\ .\end{aligned}$$

The F-dependent factors in the remaining summands can be estimated with [? , Corollary 6.1]. The factors depending on G can be estimated using Hardy's inequality

$$\|\partial_x^4 G_+\|_{L^2} \lesssim \|x\partial_x^5 G_+\|_{L^2} \lesssim \|x^2\partial_x^6 G_+\|_{L^2} \lesssim \|x^3\partial_x^7 G_+\|_{L^2} = [G_+]_{H_7}\ .$$

□

The spaces X_m^* and X_m

It is not excluded that, according to the definitions on p. 1462 (resp. formula (71)) in [?], elements of the spaces X_m^* (resp. X_m) are identified with functions only modulo a linear function $(t,x)\mapsto Cx$, $C\in\mathbb{R}$. Though main results and proofs would continue to hold with this understanding (for instance, the initial datum F_0 in Theorem 1.1 would be attained in the sense that $[F_{|t=0}-F_0]_{H_4^*}=0$), we prefer to rule out such possibility. This can be done by adding a control on the initial trace in the completion process which defines the spaces X_m^* and X_m:

$$X_m^* := \begin{cases}\text{Completion of } \left\{F\in C_c^\infty\left([0,\infty)^2\right):\ F_{|x=0}=0\right\}\\ \text{with respect to } \|\cdot\|_{X_m^*}+[\cdot_{|t=0}]_{H_2^*}\ ,\end{cases} \quad m\ge 6, \tag{35}$$

$$X_m := \begin{cases}\text{Completion of } \left\{F\in C_c^\infty\left([0,\infty)^2\right):\ F_{|x=0}=0\right\}\\ \text{with respect to } \|\cdot\|_{X_m}+[\cdot_{|t=0}]_{H_1}\ ,\end{cases} \quad m\ge 5, \tag{36}$$

where we recall that

$$\begin{aligned}\|F\|_{X_m^*} &:= \sum_{k=6}^{m}[\partial_t F]_{L^2(H_{k-4})^*}+[F]_{C^0(H_{k-2})^*}+[F]_{L^2(H_k)^*}\ ,\\ \|F\|_{X_m} &:= \sum_{k=5}^{m}[\partial_t F]_{L^2(H_{k-4})}+[F]_{C^0(H_{k-2})}+[F]_{L^2(H_k)}.\end{aligned}$$

Elements of (35) and (36) are continuous functions; in particular, their trace at $t=0$ is uniquely defined:

Lemma C.2. *For $m\ge 6$, $X_m^*\subset C^0\left([0,\infty)^2\right)$ and $F_{|t=0}\in H_{m-2}^*$ for all $F\in X_m^*$. For $m\ge 5$, $X_m\subset C^0\left([0,\infty)^2\right)$ and $F_{|t=0}\in H_{m-2}$ for all $F\in X_m$.*

Proof. In order to prove that $X_m^*\subset C^0\left([0,\infty)^2\right)$, it suffices to show that for all $T>0$ and $R>0$ there exists $C_{T,R}>0$ such that

$$\|F\|^2_{C^0((0,T)\times(0,R))} \le C\left(\|F\|^2_{X_m^*}+[F_{|t=0}]^2_{H_2^*}\right) \tag{37}$$

for all $F \in C_c^\infty\left([0,\infty)^2\right)$ with $F_{|x=0} = 0$. We have:

$$\|\partial_x F\|^2_{C^0((0,T)\times(0,\infty))} \lesssim \|\partial_x F_{|t=0}\|^2_{C^0((0,\infty))} + T\int_0^T \|\partial_t\partial_x F\|^2_{C^0((0,\infty))}dt. \tag{38}$$

It follows from Lemmas 1.2 and 1.3 in [?] that

$$\|\partial_x F_{|t=0}\|_{C^0((0,\infty))} \lesssim [F_{|t=0}]_{H_2^*}, \tag{39}$$

$$\int_0^\infty \|\partial_t\partial_x F\|^2_{C^0((0,\infty))}dt \lesssim \int_0^\infty [\partial_t F]^2_{H_2^*}dt = [\partial_t F]^2_{L^2(H_2^*)} \le [\partial_t F]^2_{L^2(H_2)^*}. \tag{40}$$

Using (39) and (40) in (38) we obtain

$$\|\partial_x F\|^2_{C^0((0,T)\times(0,\infty))} \lesssim [F_{|t=0}]^2_{H_2^*} + T\|F\|^2_{X_6^*}, \tag{41}$$

Together with $F_{|x=0} = 0$, (41) yields (37). The regularity of $F_{|t=0}$ follows from the estimates $[F]_{C^0(H^*_{m-2})} \le [F]_{C^0(H_{m-2})^*}$ (see Lemma 1.3 in [?]) and $[F]_{H_k^*} \lesssim [F]_{H_2^*} + [F]_{H^*_{m-2}}$ for $2 \le k \le m-2$ (see Corollary A.2 and Lemma 1.1 in [?]). For X_m the proof is analogous, noting that

$$\begin{aligned}\sup_{t<T}\int_0^\infty |\partial_x F|^2 dx &\lesssim \int_0^\infty |\partial_x F_{|t=0}|^2 dx + T\int_0^T\int_0^\infty |\partial_t\partial_x F|^2 dxdt \\ &\le [F_{|t=0}]^2_{H_1} + T\|F\|^2_{X_m}.\end{aligned}$$

□

With (35) and (36), all statements and proofs in [?] continue working (besides the first statement in Lemma A.6, which is discussed below). [ANYTHING ELSE NOT WORKING?] For instance, we notice that:

- (i) in the proof of Proposition 5.1[1], the sequence $\{F_h\}_{h>0} \subset X_{m+2}$ is such that $F_{h|t=0} = F_0 \in H_m \subset H_1$ for all $h > 0$, hence its limit F as $h \to 0$ belongs to X_{m+2};
- (ii) in the proof of Proposition 2.1, the sequence $\{F_\nu\}_{\nu\in\mathbb{N}} \subset X^*_{m+2}$ is such that $F_{\nu|t=0} \to F_{|t=0}$ in H_m^* (whence in H_2^*) as $\nu \to \infty$, therefore its limit F as $\nu \to \infty$ belongs to X^*_{m+2};
- (iii) in the proof of Proposition 7.1[2], the space X in which a Banach fixed point argument is applied may be equivalently defined as

  $$X = \{F \in X_6^* : \|F\|_{X_6^*} \le \delta \text{ and } F_{|t=0} = F_0\},$$

  which is a complete metric space with respect to $\|\cdot\|_{X_6^*}$.

[1] Notice a misprint: two lines above (88), $F_{|x=0} = F_0$ is to be replaced by $F_{|t=0} = F_0$.

[2] Notice a misprint: in the statement of Proposition 7.1, $F_0 \in H_6^*$ is to be replaced by $F_0 \in H_4^*$.

[ANYTHING ELSE WORTH BEING MENTIONED?]

In the first statement of Lemma A.6 of [?], we claim that $X_6^* \subset C^0([0,\infty); H_4^*)$, but in fact we only prove that $X_6^* \subset C^0([0,\infty); Y_4^*)$, where Y_4^* is a completion with respect to the $[\cdot]_{H_4^*}$-seminorm. In view of Lemma C.2, the inclusion above may now be replaced by $X_6^* \subset C^0([0,\infty)^2)$:

Lemma A.6. *$X_6^* \subset C([0,\infty)^2)$. In particular, for any $F \in X_6^*$ the trace $F|_{t=0}$ is well defined in H_4^*. In addition, for any $F \in X_6^*$ the function*

$$\phi(T) = [\partial_t F]_{L^2((0,T);H_2)^*} + [F]_{C^0((0,T);H_4)^*} + [F]_{L^2((0,T);H_6)^*} \tag{42}$$

is continuous in $[0,\infty)$ with $\phi(0) = [F|_{t=0}]_{H_4^*}$.

The inclusion $X_6^* \subset C([0,\infty)^2)$ and the regularity of the trace are already proved in Lemma C.2. Since the proof of continuity of ϕ which is currently in [?] is incomplete, we conclude by providing the full argument.

Proof. For $F \in X_6^*$, let ϕ_F be defined by (42) for $T > 0$ and by $\phi_F(0) = [F|_{t=0}]_{H_4^*}$. Obviously ϕ_F is nondecreasing in $(0,\infty)$. In addition, by a slight adaption of the proof of Lemma 1.3(i) in [?] (where the same bound is proved for $T = \infty$), we have $[F]_{C^0((0,T);H_4^*)} \le [F]_{C^0((0,T);H_4)^*}$ for all $T \in (0,\infty)$, so that

$$[F_{|t=0}]_{H_4^*} \le [F]_{C^0((0,T);H_4^*)} \le [F]_{C^0((0,T);H_4)^*} \le \phi_F(T) \qquad \text{for all } 0 < T \le \infty. \tag{43}$$

Thus ϕ_F is nondecreasing in $[0,\infty)$, whence it suffices to show that, for any $T_0 \ge 0$,

$$\limsup_{T\to T_0^+} \phi_F(T) \le \phi_F(T_0). \tag{44}$$

By density, it suffices to prove (44) for smooth functions, i.e.,

$$\limsup_{T\to T_0^+} \phi_F(T) \le \phi_F(T_0) \qquad \text{for all } F \in C_c^\infty([0,\infty)^2) \text{ with } F_{|x=0} = 0 \tag{45}$$

Indeed, assume (45) and let $F \in X_6^*$. By definition of X_6^*, for any $\varepsilon > 0$ we may find $F_\varepsilon \in C_c^\infty([0,\infty)^2)$ such that $[F_\varepsilon - F]_{X_6^*} < \varepsilon$. Taking also (43) into account, this implies that $\phi_{(F_\varepsilon - F)}(T) < \varepsilon$ for all $T \ge 0$. Therefore

$$\begin{aligned}\limsup_{T\to T_0^+} \phi_F(T) &\le \varepsilon + \limsup_{T\to T_0^+} \phi_{F_\varepsilon}(T) \overset{(45)}{\le} \varepsilon + \phi_{F_\varepsilon}(T_0) \le \varepsilon + \phi_{(F_\varepsilon - F)}(T_0) + \phi_F(T_0) \\ &\le 2\varepsilon + \phi_F(T_0),\end{aligned}$$

which implies (44) by the arbitrariness of ε.

We now show (45). Choosing F_+ and F_- as in (30), by monotone convergence we obtain

$$\begin{aligned}\limsup_{T\to T_0^+} [F]_{L^2((0,T);H_6)^*} &= \limsup_{T\to T_0^+} \int_0^\infty \left(s^{-1}[F_-(s)]^2_{L^2((0,T);H_5)} + s[F_+(s)]^2_{L^2((0,T);H_7)}\right)^{\frac12} \frac{ds}{s} \\ &= [F]_{L^2((0,T_0);H_6)^*}.\end{aligned} \tag{46}$$

By the same argument,

$$\lim_{T\to 0}[\partial_t F]_{L^2((0,T);H_2)^*} = [\partial_t F]_{L^2((0,T_0);H_2)^*}. \tag{47}$$

Therefore, it remains to prove that

$$\lim_{T\to T_0^+}[F]_{C^0((0,T);H_4)^*} \le \mathcal{G}_F(T_0) := \begin{cases} [F]_{C^0((0,T_0);H_4)^*} & \text{if } T_0 > 0 \\ [F_{|t=0}]_{H_4^*} & \text{if } T_0 = 0 \end{cases} \tag{48}$$

for all $F \in C_c^\infty([0,\infty)^2)$ with $F_{|x=0} = 0$. Let $T \le T_0 + 1$ and let $\varphi \in C_c^\infty([0,\infty))$ be a smooth cut-off function with $0 \le \varphi \le 1$ and $\varphi(t) = 1$ for $t \in [0,1]$. We define $\tilde{F} \in C_c^\infty([0,\infty)^2)$ by

$$\tilde{F}(t,x) = \begin{cases} F & \text{if } t \le T_0 \\ \varphi(T_0+t)F_{|t=T_0} & \text{if } t > T_0. \end{cases}$$

Since $\tilde{F}$ is constant in $[T_0,T]$, $[\tilde{F}]_{C^0((0,T);H_4)^*}$ can be bounded above by restricting the infimum in (4) to $F_\pm$ which are both constant in $[T_0,T]$. In addition, since $F_\pm$ are both constant in $[T_0,T]$, we have that $[F_\pm]_{C^0((0,T);H_{3\pm1})} \le [F_\pm]_{C^0([0,T_0];H_{3\pm1})}$. Therefore we obtain that $[\tilde{F}]_{C^0((0,T);H_4)^*} \le \mathcal{G}_F(T_0)$. Using that $\tilde{F} = F$ in $[0,T_0]$, by an analogous argument we also obtain that $[F-\tilde{F}]_{C^0((0,T);H_4)^*} \le [F-\tilde{F}]_{C^0((T_0,T);H_4)^*}$. Therefore

$$\begin{aligned} [F]_{C^0((0,T);H_4)^*} &\le [\tilde{F}]_{C^0((0,T);H_4)^*} + [F-\tilde{F}]_{C^0((0,T);H_4)^*} \\ &\le \mathcal{G}_F(T_0) + [F-\tilde{F}]_{C^0((T_0,T);H_4)^*}, \end{aligned}$$

where $F - \tilde{F} \in C_c^\infty([0,\infty)^2)$ satisfies $(F-\tilde{F})_{|x=0} = (F-\tilde{F})_{|t=T_0} = 0$. Hence, by translation invariance, it is enough to prove that

$$\lim_{T\to 0}[F]_{C^0((0,T);H_4)^*} = 0 \qquad \text{for all } F \in C_c^\infty([0,\infty)^2) \text{ with } F_{|t=0} = F_{|x=0} = 0. \tag{49}$$

In order to show (49), we note that

$$\|F\|_{C^0((0,T);H_4)^*} \lesssim (\|F\|_{C^0((0,T);H_3)}\|F\|_{C^0((0,T),H_5)})^{1/2}. \tag{50}$$

Indeed, for $R > 0$ we have

$$\begin{aligned} [F]_{C^0((0,T),H_4)^*} &\le \int_0^\infty \left(s^{-1}[F\chi_{(R,\infty)}(s)]^2_{C^0((0,T),H_3)} + s[F\chi_{(0,R)}(s)]^2_{C^0((0,T),H_5)}\right)^{\frac{1}{2}}\frac{ds}{s} \\ &\le \frac{1}{\sqrt{R}}[F]_{C^0((0,T),H_3)} + \sqrt{R}[F]_{C^0((0,T),H_5)} \end{aligned}$$

and (50) follows by minimization in R. Clearly, for $F \in C_c^\infty([0,\infty)^2)$ with $F_{|x=0} = F_{|t=0} = 0$, we have $\lim_{T\to 0}[F]_{C^0((0,T),H_k)} = 0$ for $k = 3,5$. Combining these limits with (50) yields (49). □

References