GROUND STATES OF A TERNARY SYSTEM INCLUDING ATTRACTIVE AND REPULSIVE COULOMB-TYPE INTERACTIONS

MARCO BONACINI AND HANS KNÜPFER

Abstract. We study a variational model where two interacting phases are embedded in a third neutral phase. The energy of the system is the sum of a local interfacial contribution and a nonlocal interaction of Coulomb type. Such models are e.g. used to describe systems of copolymer-homopolymer blends or of surfactants in water solutions. We establish existence and regularity properties of global minimizers, together with a full characterization of minimizers in the small mass regime. Furthermore, we prove uniform bounds on the potential of minimizing configurations, which in turn imply some qualitative estimates about the geometry of minimizers in the large mass regime. One key mathematical difficulty in the analysis is related to the fact that two phases have to be minimized simultaneously with both attractive and repulsive interaction present.

1. Introduction

In this work we investigate a geometric variational problem where phase separation and pattern formation are induced by the competition between interfacial energy and long-range interactions of Coulomb type (both attractive and repulsive). In particular, we focus on a *ternary system* including three different phases, two of them interacting via long-range potentials. Models of this kind, involving both interfacial energy and nonlocal interactions, appear in many physical processes; in particular, the considered model can be used to describe e.g. diblock copolymer-homopolymer blends or aqueous solutions of surfactants.

A diblock copolymer is a molecular chain consisting of two subchains made of monomers of different types, which repel each other while being chemically bonded. A complete, macroscopic segregation into two phases, which would be favored by the mutual repulsive interaction of the subchains, is forbidden by their chemical bonds: such mechanism leads to phase separation at the length scale of the polymer, commonly referred to as microphase separation. Blends of diblock copolymers with a homopolymer (a polymer chain consisting of monomers of the same species) exhibit the emergence of structures with two distinct length scales: the system undergoes a macroscopic phase separation into homopolymer- and copolymer-rich domains, and a microphase separation at the length scale of the polymer within the copolymer-rich domain. The inclusion of the third phase increases the complexity of the system; while these models show some resemblance to the pure diblock copolymer models, the observed structures are qualitatively quite different; for an account of the available results see for instance [**?**]. However, despite an extensive experimental and numerical literature, very little is known at the level of rigorous mathematical analysis.

A similar behavior is displayed by aqueous solutions of surfactants, that is amphiphilic molecules with one polar, hydrophilic "head" and a hydrophobic "tail", these two parts being linked together by a chemical bond which prevents complete phase separation. In an equilibrium solution of amphiphiles in water, the unfavorable contact between water and the apolar part of surfactant molecules leads to their self-assembly into small aggregates – *micellae* – the tails comprising the interior and the heads coating the surface. Micellar aggregates display a rich polymorphism and appear in various shapes and sizes, which share as common structural feature the fact that the width of their hydrophobic core is of the order of the amphiphile's molecular length. The preferred

Date: September 23, 2016.
2010 *Mathematics Subject Classification.* 49Q10, 35R35, 82D60.
Key words and phrases. ternary systems, copolymer-homopolymer blends, perimeter, Coulomb interaction.

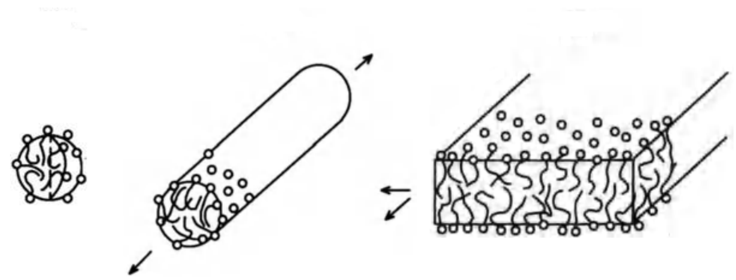


FIGURE 1. Preferred aggregation geometries of surfactants solutions: spherical micella, rod-like aggregate and planar bilayer. Reproduced from [?].

aggregation geometry can be classified in spherical micellae, rod-like aggregates and planar bilayers, see Figure 1.

In this paper we focus on a paradigmatic variational model, in which the total energy has the expression

$$E(u,v) := |Du|(\Omega) + \int_\Omega\int_\Omega G(x,y)(u-v)(x)(u-v)(y)\,\mathrm{d}x\,\mathrm{d}y\,, \tag{1.1}$$

where in general G is a Coulomb-like kernel (the Neumann Green's function of the Laplacian), and the two phase parameters $u,v \in BV(\Omega;\{0,1\})$ satisfy the constraint $uv = 0$ and identify the two interacting phases of the system as the regions where $u = 1$ and $v = 1$ respectively. A homogeneous, third phase is considered to fill the space not occupied by the u- and v-phases (the region $u = v = 0$, which will be referred to as 0-phase in this introduction).

The functional (1.1) has been derived as a generalization of the Ohta-Kawasaki energy [?] (see also [?]) for diblock copolymer systems. Indeed, in the case in which the additional constraint $u + v = 1$ excludes the presence of a third phase we recover the sharp interface limit of the Ohta-Kawasaki energy. Hence (1.1), or more general energies in which all the three possible interfaces between phases have different weights, can be used to model blends of diblock copolymers and homopolymers, see [?, ?, ?]. Also the basic characteristics of aqueous solutions of surfactants, as described above, are captured by the energy (1.1).

The main source of mathematical structure in the functional (1.1) is the presence of competing short-range attractive and long-range repulsive interactions: the first term in the energy penalizes the interfaces between the u-phase and the other regions and favors phase separation along sharp interfaces with minimal area, while the long-range Coulomb interaction term is reminiscent of the chemical bonds between the u- and the v-phase and is reduced by finely oscillating configurations.

Notice that in our model the interface between the v-phase and the 0-phase is not penalized. More in general, one can also consider models with any prescribed penalization between the three phases. We have chosen the energy (1.1) for two reasons. Firstly, the model captures essential features of the two prototype systems described above: indeed, it seems reasonable to exclude interfacial penalizations between the hydrophilic part of a surfactant and the water (corresponding to the v- and the 0-phase in the mathematical model); similarly, the same choice is appropriate for polymer blends in which the homopolymer is of the same species of one of the monomers in the diblock copolymer molecule. Also from the mathematical view point, it seems in particular intriguing to consider the case with as little interfacial energy as possible, i.e. when only the interface of one of the two phases is regularized by the inclusion of a surface energy.

The energy (1.1) is usually minimized under a constraint $\int_\Omega u = m$, which fixes the volume fraction of one of the two phases. We work in a full-space setting ($\Omega = \mathbb{R}^3$): the minimization problem in the whole space arises naturally as macroscopic limit in the low-density regime under periodic boundary conditions, see [?], and in the small volume fraction limit, see [?].

In recent years the Ohta-Kawasaki functional and its sharp interface limit have received large attention from the mathematical community. For the full-space problem, it has been shown that the energy is uniquely minimized by balls if and only if the volume fraction is below a critical threshold [?, ?, ?, ?, ?, ?], whereas a minimizer ceases to exist for large masses [?, ?], see also [?, ?] for further recent results. We also mention the papers [?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?] in which the Ohta-Kawasaki energy is studied in a bounded or periodic domain.

To date, on the contrary, only few papers deal with the ternary system (1.1). A first attempt to develop a rigorous study of the energy (1.1) is carried out by van Gennip and Peletier in [?]. Their analysis is mostly one-dimensional and provides a full characterization of global minimizers, which completes the study in [?]; however, in the higher dimensional case their best result is a lower bound which entails the linear scaling of the optimal energy in terms of the mass (see also Section 8). They also introduce a method to compute the energy of various fixed-geometry structures in terms of lower dimensional sections in the limit of large mass. In a subsequent work [?], the same authors address the study of the stability of layered structures. We also mention the paper [?], where double-bubble-like structures are considered in the context of a ternary system governed by an energy of the form (1.1) (where the nonlocal term includes also interactions with the 0-phase).

Notice that in (1.1) the v-phase contributes to the energy only through its nonlocal interaction with the u-phase and its self-repulsive interaction. In this situation we can take advantage of the results obtained by the authors together with M. Röger in [?], where we addressed the problem of minimizing the nonlocal energy

$$v \longmapsto \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} G(x,y)(u-v)(x)(u-v)(y)\,\mathrm{d}x\,\mathrm{d}y \tag{1.2}$$

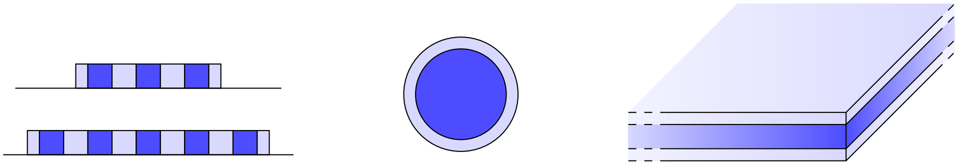
among $v \in L^1(\mathbb{R}^3;\{0,1\})$ with $uv=0$, for u fixed. One main feature of minimizing configurations v of (1.2) is that perfect *screening* is achieved: the net potential

$$\varphi(x) := \int_{\mathbb{R}^3} G(x,y)(u(y)-v(y))\,\mathrm{d}y$$

vanishes in the 0-phase and is strictly positive in the u- and v-phase. The screening property is an essential tool for some of the analysis of the full functional (1.1) carried out in this paper.

In this work, we establish existence and regularity properties of minimizers, as well as a characterization of minimizing configurations in the small mass regime. We also show some qualitative properties of minimizers in the large mass regime. In particular, we show that minimizers under the volume constraint $\int_{\mathbb{R}^3} u = m$ exist for every value $m>0$ of the mass (Theorem 3.1). An appeal to the regularity theory of *quasi-minimizers* of the area functional allows us to prove strong regularity properties of minimizing configurations (Theorem 3.2), which are enhanced by the Euler-Lagrange equations (Proposition 3.4). As a consequence of the screening property, we observe that the different connected components of a minimizer do not interact with each other (Corollary 3.3), a fact which is on the basis of the proof of the existence result. We also establish the existence of a volume threshold $m_0>0$ such that the unique minimizer for $m<m_0$ is a *spherical micella* – that is, a configuration consisting of a sphere surrounded by an annulus of the same volume (Theorem 3.5).

The question of the precise morphology of minimizing configurations for large mass m is still open. In the concluding Section 8 we collect a few results which give some qualitative information about the geometry of the minimizer. At the core of our argument lies a uniform bound on the potential φ and its gradient for minimizing configurations (Theorem 3.6). This result is analogous to the one obtained by Alberti, Choksi and Otto in [?] for the pure diblock copolymer system, and entails some qualitative properties of minimizers, such as a uniform bound on the curvature of their boundaries (Corollary 3.7) and a uniform decay estimate on large-scale density variations of the phases (Corollary 3.8). The understanding of the large mass regime is at the present stage far from being complete; the analysis in [?] shows that the functional might have a preference for lamellar structures and suggests that in the large mass limit the functional might display a behavior similar to the one considered in [?, ?] in the context of lipid bilayer membranes, and a tendency to form partially localized structures.



(a) (b) (c)

FIGURE 2. (a) In dimension 1, minimizers consist of alternating blocks of the phases u and v with equal width (except for the two end blocks), the number of blocks increasing as the total mass increases: see [**?**]. (b) Minimizers in the small mass regime are spherical micellae. (c) In the large mass regime, we expect that optimal configurations are approximately planar, lamellar structures, cut-off at large distance.

Structure of paper. In Section 2, we introduce the variational model and the main minimization problem, and we collect the statements of the main findings of the paper in Section 3. After proving some auxiliary properties in Section 4, we give the proofs of our main results in the subsequent sections: the existence result is proved in Section 5, the regularity of minimizers in Section 6 and the characterization of minimizers for small masses in Section 7. In Section 8, we eventually collect some results about the behavior of minimizers in the large mass regime.

Notation. Throughout the paper we denote by $B_\rho(x)$ the ball centered at a point $x \in \mathbb{R}^3$ with radius $\rho > 0$, writing for simplicity B_ρ for balls centered at the origin. For any measurable set $E \subset \mathbb{R}^3$ the symbol χ_E stands for its characteristic function and $|E| := \mathcal{L}^3(E)$ for its Lebesgue measure. By the notation $|E| < \infty$ we always implicitly assume that E is measurable. The symmetric difference of two measurable sets $E, F \subset \mathbb{R}^3$ is $E \triangle F := (E \backslash F) \cup (F \backslash E)$. We also write $E \Subset F$ whenever $\overline{E} \subset F$ and $\overline{E}$ is compact. We say that a sequence of measurable sets $(E_n)_n$ converges *in measure* to a set F if $|E_n \triangle F| \to 0$, and that the convergence is *local in measure* if $|(E_n \triangle F) \cap K| \to 0$ for every compact set $K \subset \mathbb{R}^3$. Sublevel sets of a function f are indicated by $\{f < \alpha\} := \{x \in \mathbb{R}^3 : f(x) < \alpha\}$, and a similar notation is used for level sets and superlevel sets. We call *universal constant* any constant which only depends on the dimension.

We recall that the *perimeter* in the sense of Caccioppoli - De Giorgi of a measurable set E in an open set $\Omega \subset \mathbb{R}^3$ is defined as the variation of its characteristic function χ_E, that is

$$\mathcal{P}(E, \Omega) := \sup \left\{ \int_E \mathrm{div}\eta \, \mathrm{d}x \; : \; \eta \in C_c^\infty(\Omega; \mathbb{R}^3), \; \|\eta\|_\infty \le 1 \right\},$$

and that $\mathcal{P}(E, \Omega) < \infty$ if and only if the distributional derivative $D\chi_E$ of χ_E is a vector-valued bounded Radon measure in Ω. In this case E is said to be a *set of finite perimeter* in Ω. We also set $\mathcal{P}(E) := \mathcal{P}(E, \mathbb{R}^3)$. For a detailed account of the theory of sets of finite perimeter, we refer the reader to the book [**?**].

2. Variational setting

The total energy. The *admissible configurations* in the variational model are represented by pairs of disjoint measurable sets (U, V) with finite measures,

$$\mathcal{A} := \left\{ (U, V) \; : \; U, V \subset \mathbb{R}^3, \; |U|, |V| < \infty, \; |U \cap V| = 0 \right\}.$$

We will often impose a volume constraint: for $m \in (0, \infty)$, we set

$$\mathcal{A}_m := \left\{ (U, V) \in \mathcal{A} \; : \; |U| = m \right\}.$$

The energy of a given configuration consists of a nonlocal Coulomb-type interaction and a term penalizing interfacial energy. Setting $w := \chi_U - \chi_V$, the energy is given by

$$\mathcal{E}(U, V) := \mathcal{P}(U) + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{w(x) w(y)}{4\pi |x - y|} \, \mathrm{d}x \, \mathrm{d}y \,. \tag{2.1}$$

Notice that only the perimeter of U appears in the energy, while the interface between V and the external phase is not penalized. The object of the paper is to investigate the minimum problem

$$e(m) := \min\left\{\mathcal{E}(U,V) \;:\; (U,V)\in\mathcal{A}_m\right\}. \tag{2.2}$$

We also introduce a notation for the nonlocal part of the energy, which is the sum of the repulsive self-interaction energy of U and V, respectively, and of the attractive mutual interaction between U and V. For $(U,V)\in\mathcal{A}$ we set

$$\mathcal{N}(U,V) := \int_U\int_U \frac{1}{4\pi|x-y|}\,\mathrm{d}x\,\mathrm{d}y + \int_V\int_V \frac{1}{4\pi|x-y|}\,\mathrm{d}x\,\mathrm{d}y - 2\int_U\int_V \frac{1}{4\pi|x-y|}\,\mathrm{d}x\,\mathrm{d}y\,, \tag{2.3}$$

so that the energy of the system can be written as $\mathcal{E}(U,V)=\mathcal{P}(U)+\mathcal{N}(U,V)$.

It is convenient to introduce the *potential* φ associated with a configuration $(U,V)\in\mathcal{A}$, defined as the unique solution to

$$\begin{cases} -\Delta\varphi = \chi_U - \chi_V\,, \\ \lim_{|x|\to\infty}|\varphi(x)| = 0, \end{cases} \tag{2.4}$$

and explicitly given by

$$\varphi(x) = \int_{\mathbb{R}^3} \frac{\chi_U(y)-\chi_V(y)}{4\pi|x-y|}\,\mathrm{d}y\,. \tag{2.5}$$

The nonlocal energy (2.3) can then be expressed in terms of the associated potential φ as

$$\mathcal{N}(U,V) = \int_{\mathbb{R}^3}\varphi\big(\chi_U-\chi_V\big)\,\mathrm{d}x = \int_{\mathbb{R}^3}|\nabla\varphi|^2\,\mathrm{d}x\,. \tag{2.6}$$

By classical elliptic regularity theory, we have $\varphi\in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^3)\cap C^{1,\alpha}(\mathbb{R}^3)$ for every $1\le p<\infty$ and $\alpha\in(0,1)$. For later reference, we also note the uniform bound

$$\sup_{\mathbb{R}^3}|\varphi| \;\le\; C\max\{|U|,|V|\}^{\frac{2}{3}} \tag{2.7}$$

for some universal $C>0$, which follows by a straightforward calculation, see e.g. [?, Lemma 3.1].

Minimization for prescribed U. In [?] the problem of the minimization of the energy $\mathcal{E}(U,\cdot)$ among measurable sets V disjoint from U, for U fixed, is studied in details. We summarize the main results of that paper in the following theorem for the reader's convenience.

Theorem 2.1 (Bonacini, Knüpfer & Röger [?])**.** *Let $U\subset\mathbb{R}^3$ be a fixed bounded measurable set. Then the minimum problem*

$$\min\left\{\mathcal{N}(U,V) \;:\; V\subset\mathbb{R}^3 \text{ with } (U,V)\in\mathcal{A}\right\} \tag{2.8}$$

has a unique (up to a set of zero Lebesgue measure) solution V_U, which satisfies $|V_U|=|U|$. Moreover, if U is open with Lipschitz boundary, then V_U is (essentially) bounded and the potential φ of the optimal configuration (U,V_U), defined in (2.4)*, satisfies the screening property*

$$\varphi\ge 0 \text{ in } \mathbb{R}^3, \qquad \varphi=0 \text{ almost everywhere in } \mathbb{R}^3\backslash(U\cup V_U). \tag{2.9}$$

One can select a precise representative of V_U in such a way that

$$U\cup V_U = \{\varphi>0\}\,. \tag{2.10}$$

Finally, for every connected component Ω of $\{\varphi>0\}$ we have $\Omega\cap U\neq\emptyset$, $\Omega\cap V_U\neq\emptyset$.

Notice that the first part of the statement is proved in [?] under the additional assumption that the set U is open, but it is straightforward to check that this condition is actually unnecessary in the proof. For later use, we introduce the following notation.

Definition 2.2 (Reduced minimizer)**.** Given an open, bounded set U with Lipschitz boundary, the unique minimizer V_U of (2.8) satisfying (2.10) is called *reduced minimizer* corresponding to U.

3. Main results

Our first result is existence of a minimizer of the energy $\mathcal{E}$ for any prescribed mass $m > 0$.

Theorem 3.1 (Existence of a minimizer)**.** *For every $m \in (0, \infty)$, there exists a solution (U, V) to the minimum problem* (2.2)*. Moreover, every minimizer satisfies $|V| = |U| = m$.*

Notice that, by virtue of Theorem 2.1, every minimizer (U, V) fulfills the volume property $|U| = |V| = m$, since V minimizes the nonlocal energy $\mathcal{N}(U, \cdot)$ among sets disjoint from U. The core of the proof is Proposition 5.1, which is a consequence of the screening of the potential, see (2.9): it is shown that the different connected components of a minimizer do not interact with each other, and they can be translated in space (as soon as that they do not overlap) without affecting the total energy, which is just the sum of the energies of the single components. With this property at hand, the existence of a minimizer can be obtained by a compactness argument. The proof of Theorem 3.1 is given in Section 5.

Our next result is concerned with regularity properties of the minimizer.

Theorem 3.2 (Structure of the minimizers)**.** *Let $(U, V) \in \mathcal{A}_m$ be a solution to the minimum problem* (2.2)*. Then the following properties hold.*

(i) *There exists $r_0 > 0$ such that*

$$\mathcal{P}(U, B_r(x)) \geq \vartheta_0 r^2, \qquad \vartheta_1 r^3 \leq |U \cap B_r(x)| \leq \vartheta_2 r^3 \tag{3.1}$$

whenever $x \in \partial U$ and $r \leq r_0$, for some universal constants $\vartheta_0, \vartheta_1 > 0$ and $\vartheta_2 < |B_1|$.

(ii) *A representative of U is open and bounded, with boundary ∂U of class C^∞.*

(iii) *The set V is the reduced minimizer corresponding to U, according to Definition 2.2. In particular, V is bounded and satisfies the regularity properties proved in* [**?**, Theorem 2.5]*.*

(iv) *There are only finitely many connected components for each of the sets U, V.*

The proof of the theorem is mainly based on the observation that the nonlocal energy can be regarded as a volume perturbation of the perimeter (Lemma 4.2). This allows us to prove that a minimizer of (2.2) is a quasi-minimizer of the perimeter (see Definition 5.2), and in turn the regularity properties stated above follow by classical theory. Such way of arguing is by now standard for proving regularity of minimizers in similar nonlocal variational problems: see, for instance, [**?**, **?**, **?**].

As a corollary, we obtain that each connected component of a minimizer is itself a minimizer of the energy with respect to its own volume. Correspondingly, it follows that the potential of minimizers vanishes outside $U \cup V$, a phenomenon we call screening.

Corollary 3.3 (Screening)**.** *Let (U, V) be the precise representative of a minimizer of problem* (2.2)*, as in Theorem 3.2. Let Ω_i be the connected components of $U \cup V$, $i = 1, \ldots, N$, and let $U_i = \Omega_i \cap U$, $V_i = \Omega_i \cap V$. Then V_i is the reduced minimizer corresponding to U_i, $|V_i| = |U_i| =: m_i$, $\mathcal{E}(U, V) = \sum_{i=1}^N \mathcal{E}(U_i, V_i)$ and each pair (U_i, V_i) is a minimizer of $\mathcal{E}$ in $\mathcal{A}_{m_i}$.*

The proofs of Theorem 3.2 and of Corollary 3.3 are given in Section 6. For reference, we also state the Euler-Lagrange equations satisfied by minimizers of (2.2). The proof is fairly standard and well-known in the case of a single self-interacting set, a sketch of the proof is given at the end of Section 6.

Proposition 3.4 (First variation)**.** *Let (U, V) be a minimizer of* (2.2)*. Then*

$$\begin{cases} H_{\partial U} + 4\varphi = \lambda & \text{on } \partial U, \\ \varphi = 0 & \text{on } \partial V \setminus \partial U, \end{cases} \tag{3.2}$$

for some $\lambda \in \mathbb{R}$. Here φ is the potential associated to (U, V), according to (2.5)*, and $H_{\partial U}$ denotes the sum of the principal curvatures of ∂U.*

For sufficiently small mass, the unique minimizer can be identified exactly. It is given by a *spherical micella*, that is a configuration where U is a ball and V is an annulus touching U from the exterior.

Theorem 3.5 (Exact minimizer for small mass)**.** *There exists a threshold $m_0 > 0$ such that for every $m \le m_0$ a minimizer of* (2.2) *is the spherical micella*

$$(U,V) = (B_{R_m}, B_{\sqrt[3]{2}R_m} \backslash B_{R_m})\,, \tag{3.3}$$

where R_m is given by $\frac{4}{3}\pi R_m^3 = m$. For $m < m_0$, the micella is the unique minimizer up to translations (and Lebesgue-negligible sets). For $m > m_0$, the micella is not a global minimizer.

The proof of the theorem, which is given in Section 7, follows the same strategy as in the proof of [**?**, Theorem 3.2], where a corresponding result is obtained for the problem with a single set U and where the minimizer is the ball. The proof relies on a perturbation argument based on the fact that in the regime of small mass m the perimeter is the leading term in the functional. By using the quantitative isoperimetric inequality and the regularity theory for quasi-minimizers of the perimeter, we first show that, if (U,V) is a minimizer, then U is close to a ball in the C^1-sense. In turn, an estimate proved by Fuglede in [**?**], combined with a careful estimate of the nonlocal energy (which is significantly more involved than in the case treated in [**?**], due to the presence of the second set V), leads to the conclusion. The proof of the fact that the set $\{m : \mathcal{E}(B_{R_m}, B_{\sqrt[3]{2}R_m} \backslash B_{R_m}) = \min_{\mathcal{A}_m} \mathcal{E}\}$ is an interval borrows an idea from [**?**].

In the regime of large mass, we expect for the minimizer that U approximates a large disc of radius R of order $m^{\frac{1}{2}}$ and thickness of order 1, enclosed by a shell of approximately uniform thickness representing the set V. While we cannot prove this conjecture, we have some partial results on the qualitative behavior of the minimizers, which we collect in Section 8. In particular, there is a uniform bound on the $C^{1,\alpha}$-norm of the potential of a minimizer.

Theorem 3.6 (Uniform bound on the potential)**.** *Let $\alpha \in (0,1)$. There exists a constant $C > 0$ such that for every $m > 0$ and for every minimizer $(U,V) \in \mathcal{A}_m$ the associated potential φ satisfies*

$$\|\varphi\|_{C^{1,\alpha}(\mathbb{R}^3)} \le C\,. \tag{3.4}$$

The proof of Theorem 3.6 is given in Section 8. By exploiting the Euler-Lagrange equations, Theorem 3.6 also yields uniform bounds on the mean curvature $H_{\partial U}$ of a minimizer.

Corollary 3.7 (Uniform bounds on curvature)**.** *Let $\alpha \in (0,1)$. There exists a constant $C > 0$ such that for every $m \ge 1$ and every minimizer $(U,V) \in \mathcal{A}_m$ we have*

$$\|H_{\partial U}\|_{C^{1,\alpha}(\partial U)} \le C\,. \tag{3.5}$$

The uniform bound on the potential in Theorem 3.6 also has certain consequences on the structure of the configuration. We first observe that as a direct consequence of the boundedness of φ and $\nabla\varphi$, the net difference of the volume for the two phases U, V averages on larger scales.

Corollary 3.8 (Averaging of the phases on large scales)**.** *There exists a constant $C > 0$ such that for every $m > 0$ and for every minimizer $(U,V) \in \mathcal{A}_m$ we have*

$$\left| \frac{1}{|B_R|} \int_{B_R(x_0)} \big(\chi_U - \chi_V\big)\,\mathrm{d}x \right| \le \frac{C}{R} \tag{3.6}$$

for every $x_0 \in \mathbb{R}^3$ and $R > 0$.

In fact, as discussed before, in the regime of large mass we expect for the minimizer that U approximates a large disc of radius R of order $m^{\frac{1}{2}}$ and thickness of order 1, enclosed by a shell of approximately uniform thickness representing the set V. For this shape, the bound (3.6) is scalingwise optimal, as can be seen by choosing an appropriate ball with radius of order $R = m^{1/2}$. On the other hand, if we replace the test function $\frac{1}{|B_R|}\chi_{B_R}$ in (3.6) by a smooth mollification kernel, one can show a higher quadratic decay, see Proposition 8.8. These estimates for the averaging on large scales are in the spirit of corresponding results for the Ohta-Kawasaki energy in a periodic setting, see [**?**, Proposition 6.1].

While Corollary 3.8 states that concentration of only a single phase of the two phases on large scales is not possible, the next result shows that the exclusive existence of a single phase is not possible on scales of length scale one. More precisely, we have the following.

Corollary 3.9. *There exists a constant $R_0 > 0$ such that for every $m > 0$ and for every minimizer $(U,V) \in \mathcal{A}_m$ we have*

$$B_R(x) \subset U \text{ or } B_R(x) \subset V \quad \text{implies} \quad R \leq R_0. \tag{3.7}$$

The proofs of Corollary 3.7, Corollary 3.8 and Corollary 3.9 are given in Section 8.

Related models and generalizations. We conclude this section with a few remarks about possible generalizations of the above model. As mentioned in the introduction, more general energies in which all the three possible interfaces between the phases are penalized (possibly with different weights) could be considered. While the analysis in [**?**] provides a detailed description in the case of dimension 1, in higher dimension the problem for general interfacial energies is still open. Notice that the inclusion of the perimeter of the phase V would add compactness properties to the system; nonetheless, the screening property on which our analysis strongly relies would no longer hold, causing the different connected components to interact and possibly leading to non-existence. However, we expect that minimizers should still exist, at least in the small mass regime where they should be close in shape to minimizers of the corresponding geometric problem without nonlocal interaction. It would be interesting as well to consider the case of anisotropic (in particular, crystalline) surface energies.

Although we have stated our results in the physically interesting case of three dimensions, we believe that the arguments used in this work can be generalized also to the case of space dimensions $n \geq 2$. Notice that for $n \geq 8$ the presence of a singular set in ∂U has to be taken into account. For $n = 2$, one has to take the representation (2.6) of the nonlocal energy, since the expression (2.3) with the logarithmic kernel is not bounded from below, as can be easily seen by considering sets (of equal volume) moving apart from each other. Also notice that, for $n = 2$, any configuration with $|U| \neq |V|$ already has infinite energy.

A further case deserving attention is when the ambient space is a bounded domain Ω and we replace the kernel $\frac{1}{4\pi|x-y|}$ by the Green's function associated with Neumann boundary conditions, or the flat torus with periodic boundary conditions. Notice that in this case, in order to have a finite energy, one has to consider the potential φ defined by $-\Delta\varphi = \chi_U - \chi_V - \lambda$, where $\lambda = |U| - |V|$. Here the condition $\lambda = 0$ seems to be equivalent to screening, since otherwise φ solves a Poisson equation in the complement of $U \cup V$ with nonzero source term.

Finally, it would be also interesting to treat more general Riesz kernels of the form $\frac{1}{|x-y|^\alpha}$, but such an extension might present additional difficulties, due to the loss of the H^1-structure; the generalization to this case would firstly require the adaptation of the arguments in [**?**] to the case in which the Laplacian is replaced by a fractional Laplacian operator.

4. Preliminaries

Before proving the main results of the present work, we recall some basic properties which will be used throughout the paper. The following lemma, analogous to [**?**, Lemma 4.5], shows how to bound the difference of the nonlocal energies of two configuration in terms of the potential of one of them.

Lemma 4.1. *For every $(U_1,V_1),(U_2,V_2) \in \mathcal{A}$ we have*

$$\mathcal{N}(U_1,V_1) - \mathcal{N}(U_2,V_2) \leq 2\int_{\mathbb{R}^3} \varphi_1\big(\chi_{U_1} - \chi_{U_2}\big)\,\mathrm{d}y - 2\int_{\mathbb{R}^3} \varphi_1\big(\chi_{V_1} - \chi_{V_2}\big)\,\mathrm{d}y\,, \tag{4.1}$$

where φ_1 denotes the potential associated with (U_1,V_1), according to (2.5).

Proof. A simple algebraic computation shows that for a symmetric, bilinear form $\beta : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ on a vector space $\mathcal{V}$, with $\beta(a,a) \geq 0$ for every $a \in \mathcal{V}$, one has

$$\beta(a,a) - \beta(b,b) = \beta(a+b, a-b) = 2\beta(a,a-b) - \beta(a-b,a-b) \leq 2\beta(a,a-b)\,. \tag{4.2}$$

We then apply the previous inequality to the bilinear form

$$\beta(f,g) := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{f(x)g(y)}{4\pi|x-y|}\,\mathrm{d}x\,\mathrm{d}y$$

defined on $L^1(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3)$, which satisfies the positivity condition $\beta(f,f)\geq 0$ by [**?**, Theorem 1.15]. Choosing $a=\chi_{U_1}-\chi_{V_1}$ and $b=\chi_{U_2}-\chi_{V_2}$ in (4.2) the conclusion follows. □

As an immediate consequence of the previous lemma and of the bound (2.7), the nonlocal part of the energy turns out to be Lipschitz-continuous with respect the L^1-distance of sets, with a constant that depends only on the volume of the sets involved.

Lemma 4.2 (Lipschitz continuity of $\mathcal{N}$)**.** *For every $m>0$ there exists a constant $c_0>0$, depending only on m, such that for every $(U_1,V_1),(U_2,V_2)\in\mathcal{A}$ with $|U_1|,|U_2|,|V_1|,|V_2|\leq m$ we have*

$$|\mathcal{N}(U_1,V_1)-\mathcal{N}(U_2,V_2)|\leq c_0\big(|U_1\triangle U_2|+|V_1\triangle V_2|\big)\,. \tag{4.3}$$

Proof. Follows from Lemma 4.1 together with the bound (2.7). □

5. Existence

In this section give the proof of Theorem 3.1. The key ingredient is contained in Proposition 5.1: it shows that different connected components of a configuration (U,V_U) do not interact with each other due to screening of the potential. Hence the connected components of a minimizing sequence can be rearranged. A technical difficulty in the proof consists in constructing a minimizing sequence with the additional property that the number of connected components and their diameters remain bounded along the sequence; this is achieved with an appeal to the regularity theory of *quasi-minimizers* of the perimeter, see Definition 5.2.

Proposition 5.1 (Screening)**.** *Let $U\subset\mathbb{R}^3$ be open, bounded and with Lipschitz boundary, and suppose that U only has finitely many connected components. Let V_U be the reduced minimizer corresponding to U, cf. Definition 2.2. Then $U\cup V_U$ has finitely many connected components. Furthermore, if Ω_i, $i=1,\dots,N$, are the connected components of $U\cup V_U$, and $U_i:=\Omega_i\cap U$, $V_i:=\Omega_i\cap V_U$, then*

$$\mathcal{N}(U,V_U)=\sum_{i=1}^N\mathcal{N}(U_i,V_i)$$

and $V_i=V_{U_i}$ is the reduced minimizer corresponding to U_i, $i=1,\dots,N$. In particular, $|V_i|=|U_i|$ and $\varphi_i=0$ on $(U_i\cup V_i)^c$, where φ_i is the potential associated to (U_i,V_i).

Proof. By Theorem 2.1, $U\cup V_U$ only has finitely many connected components. For simplicity, we present the proof only in the case when $U\cup V_U$ has two connected components, i.e. $N=2$, being straightforward to check that the conclusion continues to hold if $U\cup V_U$ has a finite number of connected components.

Notice that by Theorem 2.1 all the sets U_1, V_1, U_2, V_2 are nonempty, and clearly the sets U_i have Lipschitz boundary, for $i=1,2$. We define $\widetilde{V}_i:=V_{U_i}$ to be the reduced minimizer corresponding to U_i. Denote by φ the potential associated to (U,V_U) and by $\tilde{\varphi}_i$ the potential associated to $(U_i,\widetilde{V}_i)$. By the screening property (2.10) we have

$$U_i\cup\widetilde{V}_i=\{\tilde{\varphi}_i>0\}\,. \tag{5.1}$$

The conclusion of the proposition will follow once we prove that

$$\big(U_1\cup\widetilde{V}_1\big)\cap\big(U_2\cup\widetilde{V}_2\big)=\emptyset\,. \tag{5.2}$$

Indeed, in this case $(U,\widetilde{V}_1\cup\widetilde{V}_2)\in\mathcal{A}$ and the associated potential $\tilde{\varphi}=\tilde{\varphi}_1+\tilde{\varphi}_2$ is nonnegative and vanishes outside $U\cup\widetilde{V}_1\cup\widetilde{V}_2$. Since this screening property uniquely characterizes the reduced minimizer by [**?**, Remark 4.2], we obtain that $V_U=\widetilde{V}_1\cup\widetilde{V}_2$ and hence $V_i=\widetilde{V}_i$. Moreover,

$$\mathcal{N}(U_1\cup U_2,V_1\cup V_2)-\mathcal{N}(U_1,V_1)-\mathcal{N}(U_2,V_2)=2\int_{U_1}\tilde{\varphi}_2\,\mathrm{d}x-2\int_{V_1}\tilde{\varphi}_2\,\mathrm{d}x\,,$$

and the last term vanishes by (5.1).

Hence it remains to prove (5.2). For this, we first argue that

$$U\cup(\widetilde{V}_1\cup\widetilde{V}_2)\subset U\cup V_U\,. \tag{5.3}$$

Indeed, we have $-\Delta\tilde{\varphi}_1 \leq -\Delta\varphi$ in $U_1 \cup \widetilde{V}_1$ and $0 = \tilde{\varphi}_1 \leq \varphi$ on $\partial(U_1 \cup \widetilde{V}_1)$, which implies by the comparison principle that $\tilde{\varphi}_1 \leq \varphi$ in $U_1 \cup \widetilde{V}_1$. Hence $0 < \tilde{\varphi}_1 \leq \varphi$ in $U_1 \cup \widetilde{V}_1$ and by (2.10) we conclude that $\widetilde{V}_1 \subset U \cup V_U$. The inclusion $\widetilde{V}_2 \subset U \cup V_U$ is proved similarly.

We now show (5.2) by a contradiction argument. Indeed, if (5.2) fails, then we can find a connected component A_1 of $U_1 \cup \widetilde{V}_1$ and a connected component A_2 of $U_2 \cup \widetilde{V}_2$ with $A_1 \cap A_2 \neq \emptyset$. The set $A := A_1 \cup A_2$ is hence connected and $A \subset U \cup V_U$ by (5.3). But since A must have nonempty intersection with each of the two connected components of $U \cup V_U$, it cannot be connected: this is the desired contradiction which completes the proof of (5.2) and, in turn, of the proposition. ☐

We next define the notion of *quasi-minimizers of the perimeter functional.* The well-established regularity theory for such objects provides technical tools which will be useful in sequel. Notice that, in the literature, the term *quasi-minimizer* actually designates a larger class of sets, while sets satisfying condition (5.4) below are sometimes called (ω, r_0)-minimizers.

Definition 5.2 (Quasi-minimizer)**.** A set of finite perimeter $E \subset \mathbb{R}^n$ is said to be a *quasi-minimizer* of the perimeter if there exist positive constants $\omega > 0$ and $r_0 > 0$ such that

$$\mathcal{P}(E) \leq \mathcal{P}(F) + \omega|E \triangle F| \tag{5.4}$$

for every set of finite perimeter $F \subset \mathbb{R}^n$ with $E \triangle F \Subset B_r(x)$, for some $x \in \mathbb{R}^n$ and $r \leq r_0$.

Lemma 5.3 (Constrained minimum problem)**.** *Let $m > 0$ be fixed. There exists $R_0 > 0$ such that for any $R > R_0$ there is a solution $(U, V) \in \mathcal{A}_m$ to the constrained minimum problem*

$$\min\left\{\mathcal{E}(U, V) \ : \ (U, V) \in \mathcal{A}_m,\, U \subset B_R\right\}. \tag{5.5}$$

Furthermore, U is a quasi-minimizer of the perimeter, according to Definition 5.2, with constants ω, r_0 independent of R.

Proof. We divide the proof of the theorem into three steps.

Step 1: existence of a minimizer. The existence of a solution to (5.5) follows by an application of the direct method of the Calculus of Variations. Indeed, let (U_k, V_k) be a minimizing sequence for (5.5): since the sets $U_k \subset B_R$ have equibounded perimeter, up to selection of a subsequence we have $U_k \to U$ in measure, for some set $U \subset B_R$ of finite perimeter, see [**?**, Theorem 12.26]. Let now V_U be a minimizer of the nonlocal energy for fixed U, given by Theorem 2.1. We then have $(U, V_U) \in \mathcal{A}_m$ and, by minimality of V_U,

$$\mathcal{N}(U, V_U) \leq \mathcal{N}(U, V_k \backslash U) \tag{5.6}$$

for all $k \in \mathbb{N}$. By Lemma 4.2 and since $V_k \cap U \subset U \backslash U_k$ we have

$$\left|\mathcal{N}(U, V_k \backslash U) - \mathcal{N}(U_k, V_k)\right| \leq c_0 \left(|U \triangle U_k| + |V_k \cap U|\right) \leq 2c_o |U \triangle U_k| \to 0\,. \tag{5.7}$$

By (5.6)–(5.7) and the lower semicontinuity of the perimeter, we get

$$\mathcal{E}(U, V_U) = \mathcal{P}(U) + \mathcal{N}(U, V_U) \leq \liminf_{k\to\infty} \mathcal{P}(U_k) + \liminf_{k\to\infty} \mathcal{N}(U_k, V_k) \leq \liminf_{k\to\infty} \mathcal{E}(U_k, V_k)\,,$$

which shows that (U, V_U) is a minimizer of (5.5).

Step 2: penalized minimum problems. In order to get rid of the volume constraint, we now adapt a standard argument, see [**?**, Proposition 2.7], consisting in adding a suitable volume penalization in the functional. Precisely, we claim that we can choose $\Lambda > 0$ and $R_0 > 0$ such that for every $R > R_0$ any solution (U, V) to (5.5) is also a solution to the penalized minimum problem

$$\min\left\{J_\Lambda(U, V) := \mathcal{E}(U, V) + \Lambda\big||U| - m\big| \ : \ (U, V) \in \mathcal{A},\, U \subset B_R\right\}. \tag{5.8}$$

Notice that the existence of a minimizer of (5.8) can be established by the same argument as the one used in Step 1. We assume by contradiction that there exist sequences $R_h \to \infty$, $\Lambda_h \to \infty$ and $(U_h, V_h) \in \mathcal{A}$ minimizers of (5.8) (with Λ and R replaced by Λ_h and R_h, respectively) such that $|U_h| \neq m$ for every $h \in \mathbb{N}$.

We can assume that $|U_h| < m$ for every h (the case $|U_h| > m$ being similar). The idea is to restore the missing volume by considering variations of U_h localized in suitable balls, with a controlled increase in the perimeter; this can be done by exploiting an explicit construction by

Esposito and Fusco [?, Section 2] which was used in a similar context. The main difference here is that the variations have to be inside the balls B_{R_h}, in order to obtain admissible competitors for the minimum problem (5.8).

Notice first that the sets U_h have equibounded perimeters and $|U_h| \to m$, since $\Lambda_h \to \infty$. By [?, Proposition 2.1] we have a sequence $x_h \in \mathbb{R}^3$ such that the translated sets $U_h - x_h$ converge locally in measure to some measurable set U_0 with positive volume and finite perimeter, up to a subsequence. It is possible to find a point x_0 in the reduced boundary of the limit set U_0 such that $\operatorname{dist}(x_0, \partial B_{R_h} - x_h) > r_0 > 0$ for every h. By De Giorgi's structure theorem [?, Theorem 3.59] the rescaled sets $(U_0 - x_0)/r$ converge locally in measure as $r \to 0$ to the half space $\{x \cdot \nu_{U_0}(x_0) > 0\}$, where $\nu_{U_0}(x_0)$ denotes the generalized inner normal to U_0 at x_0. As a consequence of this convergence, given $\varepsilon > 0$ we can find $r > 0$ sufficiently small such that

$$|U_0 \cap B_{r/2}(x_r)| < \varepsilon r^3, \qquad |U_0 \cap B_r(x_r)| > C_0 r^3$$

for some universal constant $C_0 > 0$, where $x_r := x_0 - \frac{r}{2}\nu_{U_0}(x_0)$. In turn, setting $y_h := x_r + x_h$, the convergence of $U_h - x_h$ to U_0 yields

$$|U_h \cap B_{r/2}(y_h)| < \varepsilon r^3, \qquad |U_h \cap B_r(y_h)| > C_0 r^3 \tag{5.9}$$

for all h sufficiently large. By construction, assuming without loss of generality that $r < \frac{r_0}{2}$, we have $B_r(y_h) \subset B_{R_h}$, so that by modifying U_h in $B_r(y_h)$ we still get admissible competitors for problem (5.8).

The estimates (5.9) allow us to repeat the proof of [?, Proposition 2.7] and to use the bi-Lipschitz maps Φ_h, defined in that proof, to perturb the set U_h inside the ball $B_r(y_h)$: we set $\widetilde{U}_h := \Phi_h(U_h - y_h) + y_h$ and $\widetilde{V}_h := V_h \setminus \widetilde{U}_h$, so that the new pair $(\widetilde{U}_h, \widetilde{V}_h)$ is admissible in the minimum problem (5.8) and satisfies $|\widetilde{U}_h| = m$. At this point, we can proceed as in the proof of [?, Proposition 2.7], using in addition Lemma 4.2 and the obvious inequality $|V_h \triangle \widetilde{V}_h| \leq |U_h \triangle \widetilde{U}_h|$ to estimate the nonlocal term. We finally obtain

$$J_{\Lambda_h}(U_h, V_h) - J_{\Lambda_h}(\widetilde{U}_h, \widetilde{V}_h) > 0$$

for h large enough, which contradicts the minimality of (U_h, V_h) and proves the claim.

Step 3: quasi-minimality. Let now Λ and R_0 be given by the previous step, and let (U, V) satisfy the property in the statement. Consider any test set $\widetilde{U} \subset \mathbb{R}^3$ of finite perimeter with $U \triangle \widetilde{U} \Subset B_1(x)$ for some $x \in \mathbb{R}^3$. We first notice that

$$\mathcal{P}(\widetilde{U} \cap B_R) \leq \mathcal{P}(\widetilde{U}) \tag{5.10}$$

(see [?, Exercise 15.14]). Setting $\widetilde{V} := V \setminus \widetilde{U}$, we have that $(\widetilde{U} \cap B_R, \widetilde{V}) \in \mathcal{A}$ is admissible in the minimum problem (5.8): by the minimality of (U, V) for the same problem, proved in Step 2, we then have

$$\begin{aligned}
\mathcal{P}(U) &\leq \mathcal{P}(\widetilde{U} \cap B_R) + \mathcal{N}(\widetilde{U} \cap B_R, \widetilde{V}) - \mathcal{N}(U, V) + \Lambda \big| |\widetilde{U} \cap B_R| - |U| \big| \\
&\overset{(5.10)}{\leq} \mathcal{P}(\widetilde{U}) + c_0 \big(|U \triangle (\widetilde{U} \cap B_R)| + |V \triangle \widetilde{V}| \big) + \Lambda |U \triangle (\widetilde{U} \cap B_R)| \\
&\leq \mathcal{P}(\widetilde{U}) + c_0 \big(|U \triangle \widetilde{U}| + |V \cap \widetilde{U}| \big) + \Lambda |U \triangle \widetilde{U}| \\
&\leq \mathcal{P}(\widetilde{U}) + 2c_0 |U \triangle \widetilde{U}| + \Lambda |U \triangle \widetilde{U}|,
\end{aligned}$$

where we also used Lemma 4.2 in the second inequality. Hence the quasi-minimality property (5.4) follows with $\omega = 2c_0 + \Lambda$ and $r_0 = 1$. □

We now give the argument for Theorem 3.1.

Proof of Theorem 3.1. The idea of the proof is to construct a minimizing sequence (U_n, V_n) of solutions to the constrained problem (5.5) in larger and larger balls. The uniform quasi-minimality property proved in Lemma 5.3 provides uniform bounds on the number and diameters of the connected components, which allow us to conclude the proof by a compactness argument.

We fix an increasing sequence of radii $R_n \to \infty$ and we select (U_n, V_n) as a solution to the constrained minimum problem

$$\min\left\{\mathcal{E}(U,V)\ :\ (U,V)\in\mathcal{A}_m,\, U\subset B_{R_n}\right\},$$

whose existence is established in Lemma 5.3. The sequence (U_n, V_n) is clearly a minimizing sequence for the minimum problem (2.2).

The uniform quasi-minimality of U_n, proved in Lemma 5.3, ensures by classical regularity results that (a representative of) U_n is open with boundary of class $C^{1,\gamma}$ for every $\gamma\in(0,\frac12)$ and that there exists a constant $C>0$, independent of n, such that

$$|U_n\cap B_r(x)|\geq Cr^3 \qquad \text{for every } x\in\partial U_n \text{ and } r\leq r_0:=\min\{1,1/\omega\}\,, \tag{5.11}$$

see [?, Theorems 26.5, 28.1, 21.11]. In particular, U_n satisfies the assumptions of Theorem 2.1 and it is easily seen that V_n must coincide (up to a Lebesgue negligible set) with the reduced minimizer corresponding to U_n, according to Definition 2.2.

Let now $\Omega_n := U_n\cup V_n$, which coincides by (2.10) with the positivity set of the potential associated with (U_n,V_n) and is thus an open set. Notice that, by regularity of U_n, the number of its connected components is finite; hence, there are also finitely many connected components of Ω_n, since each of them has nonempty intersection with U_n. We decompose Ω_n into the union of its connected components

$$\Omega_n=\bigcup_{i=1}^{N_n}\Omega_n^{(i)}\,, \qquad \Omega_n^{(i)}=U_n^{(i)}\cup V_n^{(i)}\,,$$

where $U_n^{(i)}=\Omega_n^{(i)}\cap U_n\neq\emptyset$, $V_n^{(i)}=\Omega_n^{(i)}\cap V_n\neq\emptyset$ by Theorem 2.1. By Proposition 5.1 the interaction energy between the different connected components is zero, which means that each connected component can be translated without affecting the value of the energy, provided that the translation does not cause overlapping with other components.

From the density lower bound (5.11) it easily follows that both diameters of the connected components as well as their number are uniformly bounded, cf. [?] for similar arguments. Let $N:=\max_n N_n$ be a uniform bound on the number of connected components of Ω_n, and let $R_0:=\sup_{n,i}\operatorname{diam}\Omega_n^{(i)}$ be a uniform bound on their diameters. We consider the family of disjoint balls $B^{(k)}:=B_{R_0}(2kR_0e_1)$, $k=1,\dots,N$, and by Proposition 5.1 we can translate each connected component so that

$$\Omega_n^{(k)}\subset B^{(k)} \qquad \text{for every } k=1,\dots,N \text{ and for every } n\in\mathbb{N}$$

without affecting the total energy (where we set $\Omega_n^{(k)}=\emptyset$ if $k>N_n$). The standard compactness result [?, Theorem 12.26] yields the existence of a set of finite perimeter $U\subset\bigcup_{k=1}^N B^{(k)}$ such that upon extraction of a subsequence we have $U_n\to U$ in measure. In particular

$$\mathcal{P}(U)\leq\liminf_{n\to\infty}\mathcal{P}(U_n)\,, \qquad |U|=\lim_{n\to\infty}|U_n|=m\,. \tag{5.12}$$

Since U is a bounded measurable set, we can consider the minimizer V of the reduced minimum problem (2.8), given by Theorem 2.1. By Lemma 4.2 we get

$$|\mathcal{N}(U,V_n\backslash U)-\mathcal{N}(U_n,V_n)|\leq c_0\big(|U\triangle U_n|+|V_n\cap U|\big)\leq 2c_0|U\triangle U_n|\to 0\,. \tag{5.13}$$

Since by minimality of V we have $\mathcal{N}(U,V)\leq\mathcal{N}(U,V_n\backslash U)$ for every n, estimate (5.13) yields

$$\mathcal{N}(U,V)\leq\liminf_{n\to\infty}\mathcal{N}(U,V_n\backslash U)\leq\liminf_{n\to\infty}\big(\mathcal{N}(U_n,V_n)+2c_0|U\triangle U_n|\big)=\liminf_{n\to\infty}\mathcal{N}(U_n,V_n)\,. \tag{5.14}$$

By (5.12) and (5.14) we conclude that $(U,V)\in\mathcal{A}_m$ is a solution to (2.2). Finally, since V is the reduced minimizer corresponding to U, we have $|V|=|U|$ by Theorem 2.1. □

6. Regularity

The proof of Theorem 3.2 rests on the classical regularity theory for *quasi-minimizers of the perimeter functional*, see Definition 5.2. Indeed, since the nonlocal part of the energy behaves like a volume order term by Lemma 4.2, every solution to (2.2) is in fact a quasi-minimizer of the perimeter, and in turn enjoys the regularity properties stated in Theorem 3.2. For the proof, we use a formulation of the problem which does not include a mass constraint both in U and in V.

Proposition 6.1 (Quasi-minimality)**.** *Let $m > 0$ and let $(U_0, V_0) \in \mathcal{A}_m$ be a solution to the minimum problem* (2.2)*. Then U_0 is a quasi-minimizer of the perimeter functional, according to Definition 5.2.*

Proof. Similar to what we did in the second step of the proof of Lemma 5.3, we get rid of the volume constraint by adding a suitable penalization in the functional; in this case the proof is much simpler, since we can use a direct scaling argument.

Step 1. We claim that there exists $\Lambda > 0$ such that (U_0, V_0) is also a minimizer of

$$\min\left\{\mathcal{E}(U,V) + \Lambda\big||U| - m\big| \;:\; (U,V) \in \mathcal{A}\right\}. \tag{6.1}$$

Assume by contradiction that there exist a sequence $\Lambda_h \to \infty$ and a sequence of admissible configurations $(U_h, V_h) \in \mathcal{A}$ with $|U_h| \neq m$ for every $h \in \mathbb{N}$ such that

$$J_h(U_h, V_h) := \mathcal{E}(U_h, V_h) + \Lambda_h\big||U_h| - m\big| < J_h(U_0, V_0) = \mathcal{E}(U_0, V_0). \tag{6.2}$$

Since $\Lambda_h \to \infty$, we also have $|U_h| \to m$ as $h \to \infty$.

By an approximation argument, we can assume without loss of generality that U_h is a bounded open set with smooth boundary. Indeed, by [**?**, Theorem 13.8 and Remark 13.9] we can approximate U_h by a sequence of open and bounded sets with smooth boundary $(U_h^n)_{n\in\mathbb{N}}$ such that $U_h^n \to U_h$ in measure and $\mathcal{P}(U_h^n) \to \mathcal{P}(U_h)$ as $n \to \infty$. Setting $V_h^n := V_h \backslash U_h^n$, we have that $V_h^n \to V_h$ in measure, $(U_h^n, V_h^n) \in \mathcal{A}$ and

$$\begin{aligned} J_h(U_h^n, V_h^n) &= \mathcal{P}(U_h^n) + \mathcal{N}(U_h^n, V_h^n) + \Lambda_h\big||U_h^n| - m\big| \\ &\to \mathcal{P}(U_h) + \mathcal{N}(U_h, V_h) + \Lambda_h\big||U_h| - m\big| = J_h(U_h, V_h) \end{aligned}$$

(where the convergence of the nonlocal part of the energy follows by Lemma 4.2). Hence for n large enough $J_h(U_h^n, V_h^n) < \mathcal{E}(U_0, V_0)$, by (6.2). Replacing (U_h, V_h) by (U_h^n, V_h^n) with n sufficiently large, we can assume without loss of generality that U_h is open, bounded with smooth boundary.

By these regularity properties of U_h, the reduced minimizer V_{U_h} solving the minimum problem (2.8) corresponding to the set U_h is uniquely defined by Theorem 2.1. We can then also assume that $V_h = V_{U_h}$ and, in particular, $|V_h| = |U_h|$.

We can now conclude the proof by a scaling argument. We set

$$\lambda_h := \left(\frac{m}{|U_h|}\right)^{\frac{1}{3}} = \left(\frac{m}{|V_h|}\right)^{\frac{1}{3}},$$

which satisfy $\lambda_h \to 1$ as $h \to \infty$, and define $\widetilde{U}_h := \lambda_h U_h$, $\widetilde{V}_h := \lambda_h V_h$. Then $(\widetilde{U}_h, \widetilde{V}_h) \in \mathcal{A}_m$ and

$$\begin{aligned} \mathcal{E}(\widetilde{U}_h, \widetilde{V}_h) &= \mathcal{P}(\widetilde{U}_h) + \mathcal{N}(\widetilde{U}_h, \widetilde{V}_h) = \lambda_h^2\mathcal{P}(U_h) + \lambda_h^5\mathcal{N}(U_h, V_h) \\ &= \mathcal{E}(U_h, V_h) + \big(\lambda_h^2 - 1\big)\mathcal{P}(U_h) + \big(\lambda_h^5 - 1\big)\mathcal{N}(U_h, V_h). \end{aligned}$$

By (6.2) and in view of $||U_h| - m| = |\lambda_h^3 - 1|\,|U_h|$, this implies

$$\begin{aligned} \mathcal{E}(\widetilde{U}_h, \widetilde{V}_h) &\overset{(6.2)}{<} \mathcal{E}(U_0, V_0) + \big(\lambda_h^2 - 1\big)\mathcal{P}(U_h) + \big(\lambda_h^5 - 1\big)\mathcal{N}(U_h, V_h) - \Lambda_h\big||U_h| - m\big| \\ &= \mathcal{E}(U_0, V_0) + |\lambda_h^3 - 1|\bigg(\frac{(\lambda_h^2 - 1)}{|\lambda_h^3 - 1|}\mathcal{P}(U_h) + \frac{(\lambda_h^5 - 1)}{|\lambda_h^3 - 1|}\mathcal{N}(U_h, V_h) - \Lambda_h|U_h|\bigg) \\ &< \mathcal{E}(U_0, V_0) \end{aligned}$$

for h large enough, since $\Lambda_h \to \infty$. This contradicts the minimality of (U_0, V_0) and completes the proof of the claim.

Step 2. Let $U \subset \mathbb{R}^3$ be any set of finite perimeter with $U \triangle U_0 \Subset B_r(x)$, for some $x \in \mathbb{R}^3$ and $r \in (0,1)$. We define $V := V_0 \backslash U$ and observe that $(U,V) \in \mathcal{A}$. Notice in particular that

$$|V_0 \triangle V| = |V_0 \cap U| \leq |U_0 \triangle U| \,. \tag{6.3}$$

By minimality of (U_0,V_0) in problem (6.1) we have $\mathcal{E}(U_0,V_0) \leq \mathcal{E}(U,V) + \Lambda||U|-m|$, which yields

$$\begin{aligned} \mathcal{P}(U_0) &\leq \mathcal{P}(U) + \mathcal{N}(U,V) - \mathcal{N}(U_0,V_0) + \Lambda\big||U|-|U_0|\big| \\ &\leq \mathcal{P}(U) + c_0\big(|U_0 \triangle U| + |V_0 \triangle V|\big) + \Lambda|U_0 \triangle U| \\ &\overset{(6.3)}{\leq} \mathcal{P}(U) + \big(2c_0 + \Lambda\big)|U_0 \triangle U| \,, \end{aligned}$$

where we used Lemma 4.2 in the second inequality. This shows (5.4) with $\omega := 2c_0 + \Lambda$. □

We turn to the proof of Theorem 3.2. We will show that the regularity properties for minimizers follow by Proposition 6.1 and the classical regularity theory for quasi-minimizers of the perimeter.

Proof of Theorem 3.2. We select the precise representative of U given by its points of Lebesgue density one. Notice that this only amounts in modifying U in a Lebesgue-negligible set. By Proposition 6.1, U is a quasi-minimizer of the perimeter. Property (i) follows by application of [**?**, Theorem 21.11], with $r_0 = \min(1, \frac{1}{\omega})$.

We turn to the proof of (ii). We first note that the choice of the precise representative together with the density estimate in (i) implies that U is open. Furthermore, the lower bound in the density estimate yields boundedness of U: indeed, otherwise there exists a sequence of points $x_h \in \partial U$ such that $|x_h| \to \infty$ as $h \to \infty$, where we can assume without loss of generality that $|x_h - x_k| > 2r_0$ for every $h \neq k$. By the density estimate, we have $|U \cap B_{r_0}(x_h)| \geq \vartheta_1 r_0^3$. This immediately implies $|U| = \infty$, which is a contradiction.

By [**?**, Theorems 26.5 and 28.1] the boundary ∂U is a $C^{1,\alpha}$-surface for any $\alpha \in (0, \frac{1}{2})$, the singular set being empty since we are in three dimensions. This regularity is sufficient to show that the Euler-Lagrange equation (3.2) holds weakly on ∂U, cf. the proof of Proposition 3.4. Since ∂U is of class $C^{1,\alpha}$ and also $\varphi \in C^{1,\alpha}(\mathbb{R}^3)$, by classical Schauder estimates we obtain that ∂U is in fact of class $C^{3,\alpha}$ and the first equation of (3.2) holds strongly. Further regularity, up to C^∞, can be gained using (3.2) by a bootstrap argument as in [**?**, Proposition 2.2].

Property (iii) follows by Theorem 2.1. Finally, the regularity of U implies that it has finitely many connected components; hence there are also finitely many connected components of V, since the boundary of each of them has nonempty intersection with ∂U by [**?**, Theorem 2.4]. □

Proof of Corollary 3.3. By Theorem 3.2, there are only finitely many connected components Ω_i. By Proposition 5.1, V_i is the reduced minimizer corresponding to U_i, and furthermore $|V_i| = |U_i|$ and $\mathcal{N}(U,V) = \sum_{i=1}^N \mathcal{N}(U_i,V_i)$. In turn, since $\mathcal{P}(U) = \sum_{i=1}^N \mathcal{P}(U_i)$, we obtain $\mathcal{E}(U,V) = \sum_{i=1}^N \mathcal{E}(U_i,V_i)$. Finally, we clearly have $\mathcal{E}(U_i,V_i) = \min_{\mathcal{A}_{m_i}} \mathcal{E}$, or otherwise we could replace the pair (U_i,V_i) by a minimizer of $\mathcal{E}$ in $\mathcal{A}_{m_i}$, thus strictly decreasing the energy. □

We conclude the section with the proof of Proposition 3.4.

Proof of Proposition 3.4. We only sketch the argument for (3.2), since the computation of the first variation of the energy is quite standard, see e.g. [**?**] for the single-phase case. The proof of Theorem 3.2 shows that, for a minimizing pair (U,V), there is a representative of the minimizer such that the set U is open, bounded with boundary of class $C^{1,\alpha}$. Furthermore, V is the reduced minimizer corresponding to U. By [**?**, Theorem 2.4] U has positive distance from the set $\{\varphi = 0\}$: in particular, there is an open set A such that $U \Subset A \Subset U \cup V$. This allows us to perform a variation of the boundary of U inside the set A (hence without affecting the interface between V and the exterior). Fix any vector field

$$X \in C^1_c(A;\mathbb{R}^3) \qquad \text{with} \int_{\partial U} X \cdot \nu \,\mathrm{d}\mathcal{H}^2 = 0, \tag{6.4}$$

where ν denotes the exterior unit normal to ∂U, and consider the associated flow $(\Phi_t)_t$ defined by $\frac{\partial \Phi_t}{\partial t} = X \circ \Phi_t$, $\Phi_0(x) = x$. By a standard argument based on the implicit function theorem (see for instance [?]) we can modify the flow with a local, higher order perturbation so that the resulting variation preserves the volume of U (and, consequently, also the volume of V, by construction).

Combining the well-known formula for the first variation of the area functional (see [?]) with the expression of the first variation of the nonlocal term (which can be computed along the lines of [?] or [?]) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(\Phi_t(U),\Phi_t(V))\big|_{t=0} = \int_{\partial U} \mathrm{div}_\tau X \,\mathrm{d}\mathcal{H}^2 + 4\int_{\partial U} \varphi\, X\cdot\nu \,\mathrm{d}\mathcal{H}^2\,, \tag{6.5}$$

where div_τ denotes the tangential divergence on ∂U.

By minimality the right-hand side of (6.5) vanishes for every vector field X satisfying (6.4). Hence the first equation in (3.2) holds on ∂U in the weak sense. Furthermore, the second equation in (3.2) is an immediate consequence of the screening of the potential (2.10). □

7. The case of small mass

In this section we give the proof of Theorem 3.5, which deals with minimizers of (2.2) for small m. It is convenient to use a rescaled version of the energy: for $m>0$ and $(U,V)\in\mathcal{A}_m$, we define

$$(U_\lambda,V_\lambda) := \big(\tfrac{1}{\lambda}U, \tfrac{1}{\lambda}V\big) \in \mathcal{A}_{|B_1|} \qquad \text{with } \lambda := \left(\frac{m}{|B_1|}\right)^{\frac13},$$

and the rescaled energy

$$\mathcal{E}_\lambda(U_\lambda,V_\lambda) := \mathcal{P}(U_\lambda) + \lambda^3\,\mathcal{N}(U_\lambda,V_\lambda)\,.$$

Notice that $\mathcal{E}_\lambda(U_\lambda,V_\lambda) = \frac{1}{\lambda^2}\mathcal{E}(U,V)$. Hence the minimum problem (2.2) in the small mass regime is equivalent to the minimum problem for the rescaled energy

$$\min\big\{\mathcal{E}_\lambda(U,V) \;:\; (U,V)\in\mathcal{A}_{|B_1|}\big\} \tag{7.1}$$

for small values of the parameter λ.

In the following, we will denote by A_1 the annulus $B_{\sqrt[3]{2}}\setminus B_1$. We recall that $|A_1| = |B_1|$ and that A_1 is the reduced minimizer corresponding to B_1 by [?, Corollary A.2]. We first show that minimizers U_λ of (7.1) approximate B_1 in the limit $\lambda\to0$.

Lemma 7.1. *For every $\lambda>0$ let (U_λ,V_λ) be a solution to the rescaled minimum problem* (7.1). *Then U_λ converges in measure to the unit ball B_1 as $\lambda\to0$, up to translations.*

Proof. By the quantitative isoperimetric inequality [?], we have

$$C_0\big(\alpha(E,B_1)\big)^2 \le \mathcal{P}(E) - \mathcal{P}(B_1)\,, \qquad \text{where } \alpha(E,F) := \inf_{x\in\mathbb{R}^3} |E\,\triangle\,(x+F)|\,, \tag{7.2}$$

for every Borel set E with $|E| = |B_1|$, for some universal constant $C_0>0$. Applying (7.2) with $E=U_\lambda$ and by minimality of (U_λ,V_λ) we get

$$C_0\big(\alpha(U_\lambda,B_1)\big)^2 \le \mathcal{P}(U_\lambda) - \mathcal{P}(B_1) \le \lambda^3\big(\mathcal{N}(B_1,A_1) - \mathcal{N}(U_\lambda,V_\lambda)\big) \le C\lambda^3 \longrightarrow 0\,,$$

where in the last estimate we used the general property that the nonlocal energy $\mathcal{N}(U,V)$ of any configuration $(U,V)\in\mathcal{A}_m$ is bounded by a constant depending only on the volume of U and V: this can be easily established by the boundedness of the potential (2.7) combined with the expression of the nonlocal energy in (2.6). □

We next show that the set U_λ of the minimizing configuration is starshaped and can be described by a polar function for λ sufficiently small.

Lemma 7.2. *For $\lambda>0$ let (U_λ,V_λ) be a solution to* (7.1). *Then there is $\lambda_1>0$ such that for every $\lambda<\lambda_1$, and up to a negligible set, we have*

$$U_\lambda - x_\lambda = \{x\in\mathbb{R}^3 \;:\; |x| < 1+\psi_\lambda\big(\tfrac{x}{|x|}\big)\} \tag{7.3}$$

for some $x_\lambda\in\mathbb{R}^3$ and $\psi_\lambda\in C^{1,\alpha}(\partial B_1)$, $\alpha\in(0,\frac12)$, with $\|\psi_\lambda\|_{C^1(\partial B_1)}\to0$ as $\lambda\to0$.

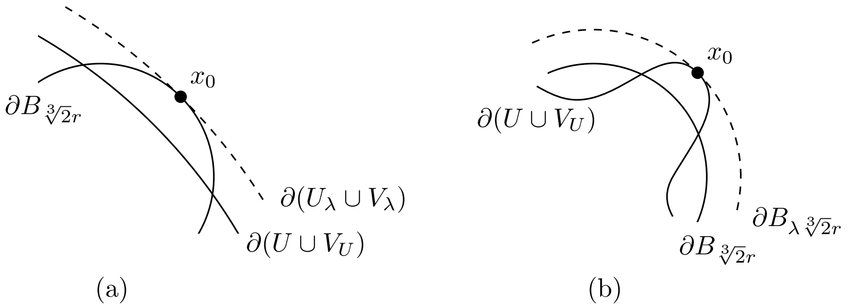


FIGURE 3. (a) Illustration for the proof of (i) of Lemma 7.3. (b) Illustration for the proof of (ii) of Lemma 7.3.

Proof. By Lemma 7.1 there are points $x_\lambda \in \mathbb{R}^3$ such that the translated sets $U_\lambda - x_\lambda$ converge in measure to B_1 as $\lambda \to 0$. Moreover, the result in Proposition 6.1 can be proved for the sets U_λ, with the quasi-minimality constants independent of λ: more precisely, there exists $\bar\lambda > 0$ such that for every $\lambda < \bar\lambda$ the set U_λ is a quasi-minimizer of the perimeter, with the constants ω, r_0 appearing in (5.4) independent of $\lambda < \bar\lambda$. In fact, this does not follow directly from the statement of Proposition 6.1, but a similar contradiction argument yields the conclusion also in the present case.

We now exploit a well-known consequence of the regularity theory for quasi-minimizers, which is essentially contained in [**?**]: it states that the convergence in measure of a uniform sequence of quasi-minimizers to a limit set with smooth boundary automatically implies the $C^{1,\alpha}$-regularity of the sets of the sequence and their convergence in the C^1-sense. In view of this result, the sets $U_\lambda - x_\lambda$ converge to B_1 also with respect to the C^1-norm, and we have the representation

$$\partial U_\lambda - x_\lambda = \{x + \psi_\lambda(x)x \ : \ x \in \partial B_1\}$$

for all λ small enough, where $\psi_\lambda \in C^{1,\alpha}(\partial B_1)$, $\alpha \in (0, \frac{1}{2})$, satisfies $\sup_\lambda \|\psi_\lambda\|_{C^{1,\alpha}(\partial B_1)} < \infty$ and $\lim_{\lambda\to 0} \|\psi_\lambda\|_{C^1(\partial B_1)} = 0$. The statement of the lemma follows. □

The following lemma allows us to get some rough information on the shape of the reduced minimizer V_U corresponding to a set U, in terms of the set U.

Lemma 7.3. *Let $U \subset \mathbb{R}^3$ be a bounded, open set with Lipschitz boundary, and let V_U be the corresponding reduced minimizer, according to Definition 2.2. Then the following properties hold.*

(i) *If $B_r \subset U$ for some $r > 0$, then $B_{\sqrt[3]{2}r} \subset U \cup V_U$.*
(ii) *If $U \subset B_r$ for some $r > 0$, then $U \cup V_U \subset B_{\sqrt[3]{2}r}$.*

Proof. We recall that the annulus $B_{\sqrt[3]{2}r} \backslash B_r$ is the reduced minimizer corresponding to B_r, see [**?**, Corollary A.2]. We prove the two properties in the statement separately.

Proof of (i). Assume by contradiction that $B_{\sqrt[3]{2}r} \backslash (U \cup V_U) \neq \emptyset$. We define

$$\lambda := \inf\bigl\{\lambda > 1 \ : \ B_{\sqrt[3]{2}r} \subset \lambda(U \cup V_U)\bigr\},$$

which satisfies $1 < \lambda < \infty$ by the contradiction assumption. We define $U_\lambda := \lambda U$, $V_\lambda := \lambda V_U$, and we observe that there exists $x_0 \in \partial B_{\sqrt[3]{2}r} \cap \partial(U_\lambda \cup V_\lambda)$, see Figure 3(a).

Denote by φ and φ_λ the potentials associated with the configurations $(B_r, B_{\sqrt[3]{2}r} \backslash B_r)$ and (U_λ, V_λ) respectively, according to (2.5). Then, in view of (2.4), (2.9) and (2.10), the map $\psi := \varphi_\lambda - \varphi$ satisfies

$$\begin{cases} -\Delta\psi \geq 0 & \text{in } B_{\sqrt[3]{2}r} = \{\varphi > 0\}, \\ \psi = \varphi_\lambda \geq 0 & \text{on } \partial B_{\sqrt[3]{2}r}, \\ \psi(x_0) = 0 = \min_{\partial B_{\sqrt[3]{2}r}} \psi \,. \end{cases}$$

Since $\psi \in C^1(\mathbb{R}^3)$, Hopf's boundary point Lemma [**?**, Lemma 3.4] yields $\nabla\psi(x_0) \cdot x_0 < 0$ and hence $\psi(x_0 + tx_0) < 0$ for t sufficiently small. This is a contradiction and proves (i).

Proof of (ii). Assume by contradiction that $(U\cup V_U)\setminus B_{\sqrt[3]{2}r}\neq\emptyset$. Similarly as before, we define

$$\lambda:=\inf\bigl\{\lambda>1\ :\ U\cup V_U\subset\lambda B_{\sqrt[3]{2}r}\bigr\}\,,$$

which satisfies $1<\lambda<\infty$ by the contradiction assumption, and we observe that there exists $x_0\in\partial B_{\lambda\sqrt[3]{2}r}\cap\partial(U\cup V_U)$, see Figure 3(b). We denote by φ and φ_λ the potentials associated with the configurations (U,V_U) and $(B_{\lambda r},B_{\lambda\sqrt[3]{2}r}\setminus B_{\lambda r})$ respectively. In view of (2.4), (2.9) and (2.10) the map $\psi:=\varphi_\lambda-\varphi$ satisfies

$$\begin{cases}-\Delta\psi\geq 0 & \text{in } U\cup V_U=\{\varphi>0\},\\ \psi=\varphi_\lambda\geq 0 & \text{on } \partial(U\cup V_U),\\ \psi(x_0)=0=\min_{\partial(U\cup V_U)}\psi\,. \end{cases}$$

Notice that the set $U\cup V_U$ satisfies an inner ball condition at x_0: indeed, the regularity result [?, Theorem 2.5] tells us that the singular points of $\partial(U\cup V_U)$ are such that the complement has zero Lebesgue density at those points, which can not happen at x_0 by construction; hence x_0 is a regular point and the interior ball condition is satisfied at x_0. Hopf's Lemma yields $\nabla\psi(x_0)\cdot\nu(x_0)<0$, where ν denotes the exterior normal to a sphere contained in $U\cup V_U$ and touching $\partial(U\cup V_U)$ at x_0. Hence $\psi(x_0+t\nu(x_0))<0$ for t sufficiently small, which is a contradiction and proves (ii). ☐

We turn to the proof of Theorem 3.5.

Proof of Theorem 3.5. For $\lambda>0$ let $(U_\lambda,V_\lambda)\in\mathcal{A}_{|B_1|}$ be a minimizer of (7.1), whose existence is established in Theorem 3.1. By Theorem 3.2, V_λ is the reduced minimizer corresponding to U_λ. The statement of Theorem 3.5 can be equivalently formulated in terms of the rescaled energy $\mathcal{E}_\lambda$: we need to show that there exists $\lambda_0>0$ such that the configuration (B_1,A_1) is a solution to (7.1) if and only if $\lambda\in(0,\lambda_0]$, and furthermore this solution is unique, up to translations and negligible sets, if $\lambda<\lambda_0$.

Step 1. In this step, we show that $(U_\lambda,V_\lambda)=(B_1,A_1)$ for all λ sufficiently small, up to translations and negligible sets. By Lemma 7.2, there exists a point x_λ such that the set $U_\lambda-x_\lambda$ has the representation (7.3), for $\lambda<\lambda_1$. Notice that we can assume without loss of generality that x_λ coincides with the barycenter of U_λ, since x_λ approximates the barycenter as $\lambda\to0$. By translation invariance, we may also assume that $x_\lambda=0$. Hence, for λ sufficiently small, the sets U_λ are *nearly-spherical*; by an estimate of Fuglede [?, ?], the polar functions ψ_λ satisfy, for some universal constant $C_0>0$, the estimates

$$\int_{\partial B_1}\bigl(\psi_\lambda^2+|\nabla\psi_\lambda|^2\bigr)\,\mathrm{d}\mathcal{H}^2\leq C_0\bigl(\mathcal{P}(U_\lambda)-\mathcal{P}(B_1)\bigr)\,,\quad \max_{\partial B_1}|\psi_\lambda|^3\leq C_0\bigl(\mathcal{P}(U_\lambda)-\mathcal{P}(B_1)\bigr)\,. \tag{7.4}$$

By minimality of (U_λ,V_λ) we have $\mathcal{E}_\lambda(U_\lambda,V_\lambda)\leq\mathcal{E}_\lambda(B_1,A_1)$ and hence

$$\mathcal{P}(U_\lambda)-\mathcal{P}(B_1)\leq\lambda^3\bigl(\mathcal{N}(B_1,A_1)-\mathcal{N}(U_\lambda,V_\lambda)\bigr)\,. \tag{7.5}$$

We now bound the right-hand side of (7.5) in terms of the H^1- and L^∞-norms of ψ_λ. We first introduce a notation for tubular neighborhoods of spheres: for $\rho>\eta>0$ we set

$$\mathcal{I}_\eta(\partial B_\rho):=B_{\rho+\eta}\setminus\overline{B}_{\rho-\eta}.$$

Setting $\eta_\lambda:=\max_{\partial B_1}|\psi_\lambda|$, we have $\eta_\lambda\to0$ as $\lambda\to0$. Moreover $B_{1-\eta_\lambda}\subset U_\lambda\subset B_{1+\eta_\lambda}$, and by Lemma 7.3 this implies

$$B_{\sqrt[3]{2}(1-\eta_\lambda)}\subset U_\lambda\cup V_\lambda\subset B_{\sqrt[3]{2}(1+\eta_\lambda)}\,,$$

see Figure 4. In turn, it follows that $A_1\triangle V_\lambda\subset\mathcal{I}_{\eta_\lambda}(\partial B_1)\cup\mathcal{I}_{\sqrt[3]{2}\eta_\lambda}(\partial B_{\sqrt[3]{2}})$. Notice also that

$$(A_1\triangle V_\lambda)\cap\mathcal{I}_{\eta_\lambda}(\partial B_1)=B_1\triangle U_\lambda\,. \tag{7.6}$$

Let φ be the potential associated with the configuration (B_1,A_1), explicitly given by

$$\varphi(x)=\begin{cases}-\frac16|x|^2+1-(\sqrt[3]{2})^{-1} & \text{if } |x|\leq1,\\ \frac16|x|^2+\frac{2}{3|x|}-(\sqrt[3]{2})^{-1} & \text{if } 1<|x|\leq\sqrt[3]{2},\\ 0 & \text{if } |x|>\sqrt[3]{2}.\end{cases}$$

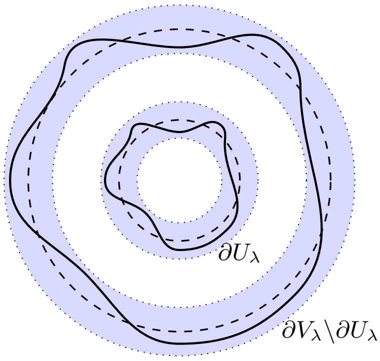


FIGURE 4. The construction in the proof of Theorem 3.5. The shaded regions are the tubular neighborhoods $\mathcal{I}_{\eta_\lambda}(\partial B_1)$ and $\mathcal{I}_{\sqrt[3]{2}\eta_\lambda}(\partial B_{\sqrt[3]{2}})$, which contain the boundaries ∂U_λ and $\partial V_\lambda \backslash \partial U_\lambda$ (represented by the solid lines) respectively.

From this expression, and with the notation $\bar\varphi := \varphi|_{\partial B_1}$, it is easily seen that

$$|\varphi(x) - \bar\varphi| \le C\big||x|-1\big| \qquad \text{and} \qquad |\varphi(x)| \le C(|x| - \sqrt[3]{2})^2 \tag{7.7}$$

for some universal constant $C > 0$. Notice that the second inequality in (7.7) follows since $\nabla\phi = 0$ on $\partial B_{\sqrt[3]{2}}$. By Lemma 4.1, and recalling (7.6), we have

$$\begin{aligned}
\mathcal{N}(B_1, A_1) - \mathcal{N}(U_\lambda, V_\lambda) &\le 2\int_{\mathbb{R}^3} \varphi\big(\chi_{B_1} - \chi_{U_\lambda}\big)\,\mathrm{d}x - 2\int_{\mathbb{R}^3} \varphi\big(\chi_{A_1} - \chi_{V_\lambda}\big)\,\mathrm{d}x \\
&\overset{(7.6)}{=} 4\int_{\mathcal{I}_{\eta_\lambda}(\partial B_1)} \varphi\big(\chi_{B_1} - \chi_{U_\lambda}\big)\,\mathrm{d}x - 2\int_{\mathcal{I}_{\sqrt[3]{2}\eta_\lambda}(\partial B_{\sqrt[3]{2}})} \varphi\big(\chi_{A_1} - \chi_{V_\lambda}\big)\,\mathrm{d}x \\
&= 4\int_{\mathcal{I}_{\eta_\lambda}(\partial B_1)} (\varphi - \bar\varphi)\big(\chi_{B_1} - \chi_{U_\lambda}\big)\,\mathrm{d}x - 2\int_{\mathcal{I}_{\sqrt[3]{2}\eta_\lambda}(\partial B_{\sqrt[3]{2}})} \varphi\big(\chi_{A_1} - \chi_{V_\lambda}\big)\,\mathrm{d}x \\
&\le 4\int_{B_1 \triangle U_\lambda} |\varphi - \bar\varphi|\,\mathrm{d}x + 2\int_{(A_1\triangle V_\lambda)\cap \mathcal{I}_{\sqrt[3]{2}\eta_\lambda}(\partial B_{\sqrt[3]{2}})} |\varphi|\,\mathrm{d}x\,.
\end{aligned}$$

Hence, using (7.7), for every λ sufficiently small we have

$$\begin{aligned}
\mathcal{N}(B_1, A_1) - \mathcal{N}(U_\lambda, V_\lambda) &\le C\int_{\partial B_1}\bigg(\int_0^{\psi_\lambda(x)} t\,\mathrm{d}t\bigg)\,\mathrm{d}\mathcal{H}^2(x) + C\eta_\lambda^2\,|\mathcal{I}_{\sqrt[3]{2}\eta_\lambda}(\partial B_{\sqrt[3]{2}})| \\
&\le C'\int_{\partial B_1} \psi_\lambda^2\,\mathrm{d}\mathcal{H}^2 + C'\eta_\lambda^3
\end{aligned} \tag{7.8}$$

where $C, C' > 0$ are positive constants. Combining (7.4), (7.5) and (7.8) we finally obtain

$$\begin{aligned}
\int_{\partial B_1} \psi_\lambda^2\,\mathrm{d}\mathcal{H}^2 + \eta_\lambda^3 &\overset{(7.4)}{\le} 2C_0\big(\mathcal{P}(U_\lambda) - \mathcal{P}(B_1)\big) \overset{(7.5)}{\le} 2C_0\lambda^3\big(\mathcal{N}(B_1, A_1) - \mathcal{N}(U_\lambda, V_\lambda)\big) \\
&\overset{(7.8)}{\le} 2C_0C'\lambda^3\bigg(\int_{\partial B_1} \psi_\lambda^2\,\mathrm{d}\mathcal{H}^2 + \eta_\lambda^3\bigg)\,.
\end{aligned}$$

This shows that $\psi_\lambda \equiv 0$ if λ is sufficiently small and hence $U_\lambda = B_1$, $V_\lambda = A_1$, as claimed.

Step 2. We set

$$\Lambda := \{\lambda > 0 \;:\; (B_1, A_1) \text{ is a minimizer of } \mathcal{E}_\lambda\} \qquad \text{and} \qquad \lambda_0 := \sup\Lambda\,.$$

By the previous step, Λ contains a non-degenerate interval of the form $(0, \bar\lambda)$, and in particular $\lambda_0 > 0$. On the other hand we have $\lambda_0 < \infty$: indeed, since the nonlocal part of the energy is the dominant term in the limit $\lambda \to \infty$, for λ large enough it is convenient to split (B_1, A_1) into two spherical components, each of mass $\frac{1}{2}|B_1|$, see also Remark 7.4. By a continuity argument, the configuration (B_1, A_1) is a minimizer of $\mathcal{E}_{\lambda_0}$.

To complete the proof it hence remains to show that for every $\lambda \in (0, \lambda_0)$ the spherical configuration (B_1, A_1) is the unique minimizer of $\mathcal{E}_\lambda$ on $\mathcal{A}_{|B_1|}$. Assume by contradiction that for some $\lambda \in (0, \lambda_0)$ there exists a minimizer $(U_\lambda, V_\lambda) \in \mathcal{A}_{|B_1|}$ of $\mathcal{E}_\lambda$ such that $|U_\lambda \triangle B_1(x)| > 0$ for all $x \in \mathbb{R}^3$. In particular

$$\mathcal{E}_\lambda(U_\lambda, V_\lambda) \le \mathcal{E}_\lambda(B_1, A_1)\,. \tag{7.9}$$

By definition of λ_0, there is $\tilde\lambda \in (\lambda, \lambda_0)$ such that (B_1, A_1) is a minimizer of $\mathcal{E}_{\tilde\lambda}$. By the isoperimetric inequality we have $\mathcal{P}(B_1) < \mathcal{P}(U_\lambda)$. Together with (7.9) this yields $\mathcal{N}(U_\lambda, V_\lambda) < \mathcal{N}(B_1, A_1)$. Hence

$$\mathcal{P}(U_\lambda) - \mathcal{P}(B_1) \overset{(7.9)}{\le} \lambda^3\big(\mathcal{N}(B_1, A_1) - \mathcal{N}(U_\lambda, V_\lambda)\big) < \tilde\lambda^3\big(\mathcal{N}(B_1, A_1) - \mathcal{N}(U_\lambda, V_\lambda)\big)\,,$$

which contradicts the minimality of (B_1, A_1) for $\mathcal{E}_{\tilde\lambda}$. This concludes the proof of the theorem. □

Remark 7.4. It is straightforward to compute the exact value of the energy of a spherical micella $(U, V) = (B_{R_m}, B_{\sqrt[3]{2}R_m} \backslash B_{R_m})$ of mass m, as in (3.3): indeed, one has

$$\mathcal{E}(B_{R_m}, B_{\sqrt[3]{2}R_m} \backslash B_{R_m}) = 4\pi R_m^2 + 8\pi\Big(\frac{1}{3} - \frac{2^{2/3}}{5}\Big)R_m^5 = c_1 m^{\frac{2}{3}} + c_2 m^{\frac{5}{3}}\,,$$

with $c_1 = (36\pi)^{\frac{1}{3}}$, $c_2 = 6(\frac{4\pi}{3})^{-\frac{2}{3}}(\frac{1}{3} - \frac{2^{2/3}}{5})$. More in general, the energy of a configuration $(U, V) \in \mathcal{A}_m$ consisting of N disjoint micellae of equal mass m/N is just the sum of the energy of the single components, that is, $\mathcal{E}(U, V) = c_1 N^{\frac{1}{3}} m^{\frac{2}{3}} + c_2 N^{-\frac{2}{3}} m^{\frac{5}{3}}$. The energy of a micella of mass

$$\tilde m_0 := \frac{c_1(2^{1/3} - 1)}{c_2(1 - 2^{-2/3})}$$

is is hence equal to the energy of two disjoint micellae of mass $\tilde m_0/2$. In particular, $\tilde m_0$ provides an upper bound on the critical threshold m_0.

8. The large mass regime

In this final section we collect some considerations about the behavior of the minimizers of the energy (2.1) in the large mass regime. We first recall that, by a result in [**?**], the minimal energy scales linearly with respect to the mass for large values of m.

Proposition 8.1 (van Gennip & Peletier [**?**])**.** *There exist universal constants $C_0, C_1 > 0$ such that*

$$C_0 \max\{m^{\frac{2}{3}}, m\} \le e(m) \le C_1 \max\{m^{\frac{2}{3}}, m\}\,. \tag{8.1}$$

The upper bound in (8.1) follows by the construction of explicit examples: consider, for instance, the energy of a single spherical micella for $m \le 1$ and the energy of N disjoint micellae, with $N \sim m$, for $m > 1$ (see Remark 7.4). The lower bound is proved, by means of an interpolation inequality, in [**?**, Theorem 7].

We next give a simple lemma on the regularity of the minimal energy function (2.2).

Lemma 8.2. *The minimal energy $e : (0, \infty) \to (0, \infty)$, defined in (2.2), is monotonically increasing and Lipschitz continuous on $[m_0, \infty)$, for every $m_0 > 0$.*

Proof. The fact that e is monotonically increasing follows from a straightforward scaling argument. We turn to the proof of the Lipschitz continuity of e on $[m_0, \infty)$ for fixed $m_0 > 0$. Indeed, for $m \ge m_0$, let $(U, V) \in \mathcal{A}_m$ be a minimizer of $\mathcal{E}$ with mass m. For $m' \in (m, 2m)$, we set $\lambda = (m'/m)^{\frac{1}{3}}$ and consider the rescaled configuration $(\lambda U, \lambda V) \in \mathcal{A}_{m'}$. We then have

$$e(m') \le \mathcal{E}(\lambda U, \lambda V) = \mathcal{E}(U, V) + (\lambda^2 - 1)\mathcal{P}(U) + (\lambda^5 - 1)\mathcal{N}(U, V) \le e(m)\bigg(1 + \frac{C(m' - m)}{m}\bigg)$$

for some constant $C > 0$. Together with (8.1), this yields

$$e(m') - e(m) \le C\frac{e(m)}{m}(m' - m) \le \widetilde C(m' - m)$$

for all $m \ge m_0$ and $m' \in (m, 2m)$, with a constant $\widetilde C$ depending also on m_0. The Lipschitz continuity on $[m_0, \infty)$ follows. □

We next consider the minimal energy per unit mass: we show that the optimal energy-to-mass ratio is in fact attained in the limit as $m \to \infty$.

Proposition 8.3. *We have*

$$f^* := \inf_{m>0} \frac{e(m)}{m} = \lim_{m\to\infty} \frac{e(m)}{m}.$$

Proof. The proof follows by constructing suitable test configurations, see e.g. [?, Theorem 4.15] for a similar argument. We set $f(m) := \frac{e(m)}{m}$. For $\varepsilon > 0$ given, let $m_\varepsilon > 0$ be such that $f(m_\varepsilon) \leq f^* + \varepsilon$, and let $(U_\varepsilon, V_\varepsilon) \in \mathcal{A}_{m_\varepsilon}$ be a minimizer of the energy. Let also $k_\varepsilon \in \mathbb{N}$ be such that $\varepsilon k_\varepsilon \geq 1$.

For every mass $m > k_\varepsilon m_\varepsilon$ we can write $m = k m_\varepsilon + \tilde{m}$, where $k \in \mathbb{N}$, $k \geq k_\varepsilon$, and $\tilde{m} \in [0, m_\varepsilon)$. If $(\tilde{U}, \tilde{V}) \in \mathcal{A}_{\tilde{m}}$ is a minimizer for prescribed mass $\tilde{m}$, we have

$$\mathcal{E}(\tilde{U}, \tilde{V}) = e(\tilde{m}) \overset{(8.1)}{\leq} C_1 \max\{\tilde{m}^{2/3}, \tilde{m}\} \leq C m_\varepsilon \,, \tag{8.2}$$

where the last inequality follows by observing that m_ε is bounded from below by a positive constant, as a consequence of the fact that $\lim_{m\to 0} f(m) = \infty$ by (8.1).

We then consider a configuration $(U, V) \in \mathcal{A}_m$ given the union of k copies (not overlapping) of $(U_\varepsilon, V_\varepsilon)$ with one copy of $(\tilde{U}, \tilde{V})$. The estimate

$$f(m) \leq \frac{1}{m}\Big(k\mathcal{E}(U_\varepsilon, V_\varepsilon) + \mathcal{E}(\tilde{U}, \tilde{V})\Big) \overset{(8.2)}{\leq} \frac{k e(m_\varepsilon)}{m} + \frac{C m_\varepsilon}{m} \leq f(m_\varepsilon) + \frac{C}{k} \leq f^* + \varepsilon + C\varepsilon$$

concludes the proof. □

It is an open question whether the minimal value f^* is attained also at a finite mass. In [?] the energy per unit mass of structures with different geometries is computed. In particular, one-dimensional lamellar structures achieve the value $(\frac{9}{2})^{\frac{1}{3}}$ in the limit of large mass; such structures are a candidate for the preferred morphology, while it seems that spherical structures have an higher energy-per-mass ratio.

We conclude the paper by establishing uniform upper bounds on the potential φ and on its gradient $\nabla\varphi$ for minimizing configurations of arbitrary volume. We also deduce some further consequences of these estimates. We will use the following consequence of Harnack's inequality.

Lemma 8.4. *Let $x_0 \in \mathbb{R}^3$ and suppose that $\varphi \in W^{2,p}(B_{2R}(x_0))$ satisfies $\varphi \geq 0$ and $|\Delta\varphi| \leq 1$. Then for every $x \in B_R(x_0)$ we have the bound*

$$\alpha\varphi(x_0) - \tilde{\alpha}R^2 \leq \varphi(x) \leq \beta\varphi(x_0) + \tilde{\beta}R^2 \tag{8.3}$$

for some universal constants $\alpha, \tilde{\alpha}, \beta, \tilde{\beta} > 0$.

Proof. We define φ_1 as the solution to

$$\begin{cases} \Delta\varphi_1 = \Delta\varphi & \text{in } B_{2R}(x_0), \\ \varphi_1 = 0 & \text{on } \partial B_{2R}(x_0). \end{cases}$$

We also define $\psi(x) := \frac{2}{3}R^2 - \frac{1}{6}|x - x_0|^2$. Since $\varphi_1 = \psi = 0$ on $\partial B_{2R}(x_0)$ and $|\Delta\varphi_1| \leq |\Delta\psi| = 1$, by the maximum principle we have $|\varphi_1| \leq |\psi|$. In turn, this implies

$$\frac{1}{6}|x - x_0|^2 - \frac{2}{3}R^2 \leq \varphi_1(x) \leq -\frac{1}{6}|x - x_0|^2 + \frac{2}{3}R^2 \qquad \text{in } B_{2R}(x_0). \tag{8.4}$$

Moreover, the function $\varphi_2 := \varphi - \varphi_1$ solves

$$\begin{cases} -\Delta\varphi_2 = 0 & \text{in } B_{2R}(x_0), \\ \varphi_2 = \varphi & \text{on } \partial B_{2R}(x_0). \end{cases}$$

Since φ is nonnegative by assumption, we have $\varphi_2 \geq 0$ in $B_{2R}(x_0)$. Hence by Harnack's inequality there are universal constants $\alpha \in (0, 1)$, $\beta > 1$ such that for every $x \in B_R(x_0)$

$$\alpha\varphi_2(x_0) \leq \varphi_2(x) \leq \beta\varphi_2(x_0) \,. \tag{8.5}$$

It follows that for $x \in B_R(x_0)$

$$\varphi_2(x) \overset{(8.5)}{\geq} \alpha\varphi_2(x_0) = \alpha\big(\varphi(x_0) - \varphi_1(x_0)\big) \overset{(8.4)}{\geq} \alpha\varphi(x_0) - \frac{2}{3}\alpha R^2 \,. \tag{8.6}$$

In turn, by (8.4) and (8.6), we obtain for every $x \in B_R(x_0)$

$$\varphi(x) = \varphi_1(x) + \varphi_2(x) \geq \frac{1}{6}|x - x_0|^2 - \frac{2}{3}R^2 + \alpha\varphi(x_0) - \frac{2}{3}\alpha R^2 \geq \alpha\varphi(x_0) - \frac{2}{3}(\alpha + 1)R^2 \,,$$

which yields the lower bound in (8.3). The upper bound follows similarly from (8.4) and (8.5). □

The proof of Theorem 3.6 follows directly from the following two lemmas.

Lemma 8.5 (Bound on φ)**.** *For every $m > 0$ and for every minimizer $(U, V) \in \mathcal{A}_m$ of* (2.2)*, the associated potential φ, given by* (2.4)*, satisfies*

$$\|\varphi\|_{L^\infty(\mathbb{R}^3)} \leq C$$

for some universal constant $C > 0$.

Proof. Let $(U, V) \in \mathcal{A}_m$ be a minimizer of the energy, and let $\bar{x}$ be a maximum point of φ. By translation invariance we may assume that $\bar{x} = 0$. By the maximum principle, we have $0 \in \overline{U}$.

We first claim that we may assume

$$B_1 \subset U \cup V \,. \tag{8.7}$$

Indeed, otherwise by (2.9) there is $x_0 \in B_1$ with $\varphi(x_0) = 0$. An application of Lemma 8.4 to φ in $B_{2|x_0|}(x_0)$ then yields $\varphi(0) \leq \tilde{\beta}$, which provides a uniform upper bound on $\varphi(0)$. Hence, in the following we will assume (8.7).

We next show that we can assume without loss of generality that

$$|U \cap B_1| \geq C_0 \tag{8.8}$$

for a universal constant $C_0 > 0$. Indeed, let $\varepsilon := |U \cap B_1|$. We decompose $\varphi = \varphi_1 + \varphi_2$, where φ_1 and φ_2 are given as the solutions to

$$\begin{cases} -\Delta\varphi_1 = \chi_{U\cap B_1} - \chi_{V\cap B_1}, \\ \lim_{|x|\to\infty} |\varphi_1(x)| = 0, \end{cases} \qquad \text{and} \qquad \begin{cases} -\Delta\varphi_2 = \chi_{U\setminus B_1} - \chi_{V\setminus B_1}, \\ \lim_{|x|\to\infty} |\varphi_2(x)| = 0. \end{cases}$$

By the maximum principle φ_2 attains its maximum at some point $x_0 \in \overline{U}\backslash B_1$. In particular, we have $\varphi_2(0) \leq \varphi_2(x_0)$ and hence

$$\varphi(0) - \varphi(x_0) \leq \varphi_1(0) - \varphi_1(x_0) \,. \tag{8.9}$$

Let $r_\varepsilon > 0$ denote the radius of a ball of volume ε. By classical rearrangement inequalities [**?**, Theorem 3.4], since $|V \cap B_1| = |B_1| - \varepsilon$ by (8.7), we have

$$\varphi_1(0) - \varphi_1(x_0) \leq \frac{1}{4\pi}\bigg(\int_{B_{r_\varepsilon}} \frac{1}{|y|}\,\mathrm{d}y - \int_{B_1\setminus B_{r_\varepsilon}} \frac{1}{|y|}\,\mathrm{d}y\bigg) + \frac{1}{4\pi}\int_{B_1} \frac{1}{\left|\frac{x_0}{|x_0|} - y\right|}\,\mathrm{d}y = \Big(r_\varepsilon^2 - \frac{1}{2}\Big) + \frac{1}{3} \,. \tag{8.10}$$

By (8.9)–(8.10) and for ε sufficiently small, we hence get $\varphi(0) < \varphi(x_0)$, which contradicts our assumption that φ attains its maximum at 0. In the following, we may hence assume (8.8).

We now conclude the proof with the construction of a suitable competitor, obtained by removing a ball around the origin from U and by moving it to a large distance. The set $U \cup V$ is bounded, i.e. $U \cup V \subset B_{R_0}$ for some $R_0 > 0$. Now, let $D := U \cap B_1$ and let $D + Re_1$ be a translated copy of D centered at a point at distance $R > R_0 + 1$ from the origin. We set

$$\widetilde{U} := (U\backslash D) \cup (D + Re_1).$$

In particular, we have $(\widetilde{U}, V) \in \mathcal{A}_m$. By minimality of (U, V), we have

$$\begin{aligned}0 &\geq \mathcal{E}(U,V) - \mathcal{E}(\widetilde{U},V) = \mathcal{P}(U) - \mathcal{P}(\widetilde{U}) + \mathcal{N}(U,V) - \mathcal{N}(\widetilde{U},V)\\ &\geq -2\mathcal{P}(B_1) + 2\int_D\int_{\mathbb{R}^3} \frac{(\chi_{U\setminus D} - \chi_V)(x)}{4\pi|x-y|}\,\mathrm{d}x\,\mathrm{d}y - 2\int_{D+Re_1}\int_{\mathbb{R}^3} \frac{(\chi_{U\setminus D} - \chi_V)(x)}{4\pi|x-y|}\,\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

and hence

$$\int_D \varphi\,\mathrm{d}x \leq \mathcal{P}(B_1) + \int_D \varphi_3\,\mathrm{d}x + \int_{D+Re_1} (\varphi - \varphi_3)\,\mathrm{d}x\,, \tag{8.11}$$

where φ_3 is the potential assigned to the set $D = U \cap B_1$, i.e.

$$\begin{cases} -\Delta\varphi_3 = \chi_D, \\ \lim_{|x|\to\infty} |\varphi_3(x)| = 0. \end{cases}$$

For any $\varepsilon > 0$, we can choose $R > R_0$ sufficiently large such that the third term on the right hand side of (8.11) is smaller than ε in modulus. Furthermore, we have $|\varphi_3| \leq C$ for some universal constant $C > 0$. Finally, by an application of Lemma 8.4, we have $\varphi(x) \geq \alpha\varphi(0) - \tilde{\alpha}$ for all $x \in B_1$. Inserting these estimates into (8.11) and recalling that the volume of D is uniformly bounded from below by (8.8), we obtain the desired bound $\varphi(0) \leq C$ for a universal constant C. □

Lemma 8.6 (Bound on $\nabla\varphi$)**.** *For every $m > 0$ and for every minimizer $(U, V) \in \mathcal{A}_m$ of (2.2), the associated potential φ, given by (2.4), satisfies*

$$\|\nabla\varphi\|_{C^{0,\alpha}(\mathbb{R}^3)} \leq C$$

for $\alpha \in (0, 1)$, for some constant $C > 0$ depending only on α.

Proof. Fix any point $x_0 \in \mathbb{R}^3$ and let $\varphi = \varphi_1 + \varphi_2$, where φ_1 and φ_2 are the solutions to

$$\begin{cases} -\Delta\varphi_1 = (\chi_U - \chi_V)\chi_{B_1(x_0)}, \\ \lim_{|x|\to\infty} |\varphi_1(x)| = 0, \end{cases} \qquad \text{and} \qquad \begin{cases} -\Delta\varphi_2 = (\chi_U - \chi_V)\chi_{B_1^c(x_0)}, \\ \lim_{|x|\to\infty} |\varphi_2(x)| = 0. \end{cases}$$

A straightforward calculation using (2.5) then yields, for $\alpha \in (0, 1)$,

$$\|\varphi_1\|_{C^{1,\alpha}(\mathbb{R}^3)} \leq C \tag{8.12}$$

for some constant $C > 0$ depending only on α. Moreover, φ_2 is harmonic in $B_1(x_0)$ and by interior estimates [**?**, Theorem 2.10] we have

$$\sup_{B_{1/2}(x_0)} \Big(|\nabla\varphi_2| + |D^2\varphi_2|\Big) \leq C \sup_{B_1(x_0)} |\varphi_2| \leq C\left(\|\varphi\|_{L^\infty(\mathbb{R}^3)} + \|\varphi_1\|_{L^\infty(\mathbb{R}^3)}\right) \overset{(8.12)}{\leq} C \tag{8.13}$$

for some universal constant $C > 0$, where we also used Lemma 8.5 in the last inequality. Hence using (8.12) we have $|\nabla\varphi(x_0)| \leq |\nabla\varphi_1(x_0)| + |\nabla\varphi_2(x_0)| \leq C$, which shows that $\|\nabla\varphi\|_{L^\infty(\mathbb{R}^3)}$ is uniformly bounded since x_0 is arbitrary. In turn, also $\|\nabla\varphi_2\|_{L^\infty(\mathbb{R}^3)}$ is uniformly bounded.

Furthermore, by the bound on $|D^2\varphi_2|$ in (8.13) we have

$$|\nabla\varphi_2(x) - \nabla\varphi_2(x_0)| \leq C|x - x_0|^\alpha \qquad \text{for all } x \in B_{1/2}(x_0)\,.$$

This, combined with the uniform bound on $\|\nabla\varphi_2\|_{L^\infty(\mathbb{R}^3)}$, yields $\|\nabla\varphi_2\|_{C^{0,\alpha}(\mathbb{R}^3)} \leq C$. In turn, by using (8.12) we obtain $\|\nabla\varphi\|_{C^{0,\alpha}(\mathbb{R}^3)} \leq C$. □

Proof of Theorem 3.6. Follows from Lemma 8.5 and Lemma 8.6. □

The following auxiliary lemma is used in the proof of Corollary 3.7.

Lemma 8.7. *Let $U \subset \mathbb{R}^n$ be open, bounded with boundary of class C^2. Assume that the mean curvature $H_{\partial U}$ of ∂U is bounded from below by a constant $H_0 > 0$. Then*

$$\mathcal{P}(U) \geq H_0|U|. \tag{8.14}$$

Proof. Let h_U be the Cheeger constant of U, defined as

$$h_U := \min_{F\subset U} \frac{\mathcal{P}(F)}{|F|}\,, \tag{8.15}$$

and let $E \subset U$ be a Cheeger set in U, that is a solution to the minimum problem (8.15), which exists by standard theory. It is hence sufficient to show that $h_U \geq H_0$, since in this case

$$\frac{\mathcal{P}(U)}{|U|} \geq \frac{\mathcal{P}(E)}{|E|} = h_U \geq H_0\,.$$

We first recall that $\partial E \cap U$ is analytic except for a singular set Σ with Hausdorff dimension at most $n-8$, and $(\partial E\backslash\Sigma)\cap U$ has constant mean curvature equal to h_U. Furthermore, ∂E is of class C^1 in a neighborhood of any intersection point $x_0 \in \partial E \cap \partial U$ and the two boundaries meet tangentially, see e.g. [**?**].

By a simple scaling argument, we have $\partial E \cap \partial U \neq \emptyset$. By a suitable choice of coordinates we may hence assume that $0 \in \partial E \cap \partial U$ and that there is an open neighborhood $A = A' \times (-a, a)$ of 0, with $A' \subset \mathbb{R}^{n-1}$ and $a > 0$, such that $U \cap A$ and $E \cap A$ can be represented as the subgraphs of two functions $u \in C^2(A')$ and $v \in C^1(A')$ respectively, with

$$-a \leq v \leq u \leq a \text{ in } A', \qquad u(0) = v(0) = 0, \qquad |\nabla u(0)| = |\nabla v(0)| = 0\,.$$

Observe that E minimizes the functional $F \mapsto \mathcal{P}(F) - h_U|F|$ among subsets $F \subset U$ with finite perimeter. By taking inner variations we easily obtain that

$$\int_{A'} \frac{\nabla v\cdot\nabla w}{\sqrt{1+|\nabla v|^2}}\,\mathrm{d}x \leq h_U \int_{A'} w\,\mathrm{d}x \qquad \text{for all } w \in C^1_c(A') \text{ with } w \geq 0\,. \tag{8.16}$$

Estimate (8.16) implies that the mean curvature $H_{\partial E}$ of ∂E is $\mathcal{H}^{n-1}$-a.e. in $\partial E \cap A$ well-defined and satisfies

$$h_U \geq H_{\partial E} \qquad \mathcal{H}^{n-1}\text{-a.e. on } \partial E \cap A\,. \tag{8.17}$$

If $\mathcal{H}^{n-1}(\partial E \cap \partial U) > 0$, then $H_{\partial E} = H_{\partial U}$ for $\mathcal{H}^{n-1}$-a.e. point in $\partial E \cap \partial U$. In view of (8.17) this yields $h_U \geq H_0$, as claimed. Otherwise, if $\mathcal{H}^{n-1}(\partial E \cap \partial U) = 0$, then the equality $H_{\partial E} = h_U$ holds $\mathcal{H}^{n-1}$-a.e. in $A \cap \partial E$ and yields

$$-\mathrm{div}\Big(\frac{\nabla v}{\sqrt{1+|\nabla v|^2}}\Big) = h_U \qquad \mathcal{H}^{n-1}\text{-a.e. in } A'.$$

This equation improves the regularity of v, which is in particular of class C^2, see e.g. [**?**, Section 7.7] and [**?**, Theorem 9.19]. Since $u - v$ has a minimum in 0, we obtain $\Delta u(0) \geq \Delta v(0)$ which in turn implies $h_U = H_{\partial E}(0) \geq H_{\partial U}(0) \geq H_0$. This concludes the proof of the lemma. □

Proof of Corollary 3.7. By Proposition 3.4 the identity

$$H_{\partial U} + 4\varphi = \lambda \tag{8.18}$$

holds on ∂U for some constant $\lambda \in \mathbb{R}$. If the curvature $H_{\partial U}$ is uniformly bounded from below by a positive constant $H_0 > 0$, by Lemma 8.7 we have $\mathcal{P}(U) \geq H_0|U|$, which is impossible for large H_0 by Proposition 8.1. Hence for every minimizer we can find a point $x_0 \in \partial U$ such that $H_{\partial U}(x_0)$ is uniformly bounded. By evaluating (8.18) at x_0 and using the boundedness of φ proved in Theorem 3.6 we obtain a uniform upper bound on the Lagrange multiplier λ, independent of the minimizer. In turn, by (8.18) it follows that $\|H_{\partial U}\|_\infty \leq 4\|\varphi\|_\infty + |\lambda|$, which yields the uniform bound on $H_{\partial U}$. The bound on the tangential gradient $\nabla_\tau H_{\partial U}$ in $C^{0,\alpha}$ follows by differentiating tangentially (8.18) on ∂U and using again Theorem 3.6. □

We next turn to the proof of Corollary 3.8. In fact, we will prove two related versions which both show net averaging of U, V on large scales. Let $\eta : \mathbb{R}^3 \to \mathbb{R}$ be a standard mollification kernel with $\eta \geq 0$, $\int_{\mathbb{R}^3} \eta = 1$ and $\mathrm{supp}\,\eta \subset B_1$ and let

$$\eta_R(x) := \frac{1}{R^3}\,\eta\Big(\frac{x}{R}\Big)$$

for $R > 0$. With this notation, we have the following result.

Proposition 8.8. *There exists a constant $C > 0$ such that for every $m > 0$ and for every minimizer $(U, V) \in \mathcal{A}_m$ we have*

$$\left| \int_{B_R(x_0)} \eta_R(\cdot - x_0)\big(\chi_U - \chi_V\big)\,\mathrm{d}x \right| \le \frac{C}{R^2} \tag{8.19}$$

for every $x_0 \in \mathbb{R}^3$ and $R > 0$. Furthermore,

$$\left| \frac{1}{|B_R|} \int_{B_R(x_0)} \big(\chi_U - \chi_V\big)\,\mathrm{d}x \right| \le \frac{C}{R}\,. \tag{8.20}$$

Proof. Without loss of generality, we may assume $x_0 = 0$. The estimate (8.19) is easily established by integrating by parts two times in the integral: indeed, we have

$$\left| \int_{\mathbb{R}^3} \eta_R\big(\chi_U - \chi_V\big)\,\mathrm{d}x \right| = \left| \int_{\mathbb{R}^3} \eta_R \Delta\varphi\,\mathrm{d}x \right| = \left| \int_{\mathbb{R}^3} \varphi \Delta\eta_R\,\mathrm{d}x \right| \overset{(3.4)}{\le} \frac{C}{R^2}\,.$$

This yields (8.19). The estimate (8.20) is obtained similarly integrating by parts just once. □

Proof of Corollary 3.8. Follows from Proposition 8.8. □

Proof of Corollary 3.9. Assume that $B_R(x_0) \subset U$ for some minimizer $(U, V) \in \mathcal{A}_m$. Let φ the potential associated to (U, V) and define the function $w(x) := \varphi(x) + \frac{1}{6}|x - x_0|^2$. Then w is harmonic in $B_R(x_0)$ and by the minimum principle

$$\varphi(x_0) = w(x_0) \ge \min_{\partial B_R(x_0)} w = \min_{\partial B_R(x_0)} \varphi + \frac{R^2}{6} \ge \frac{R^2}{6}\,.$$

Since $\varphi(x_0)$ is uniformly bounded by Theorem 3.6, we obtain a uniform bound on R. In the case $B_R(x_0) \subset V$ the conclusion follows similarly by considering $w(x) := \varphi(x) - \frac{1}{6}|x - x_0|^2$. □

Acknowledgments. MB is member of the INdAM - GNAMPA Project 2015 “Critical Phenomena in the Mechanics of Materials: a Variational Approach”. MB and HK would like to thank the Mathematics Center Heidelberg (MATCH) for support.

(Marco Bonacini) Institut für Angewandte Mathematik, Universität Heidelberg, Im Neuenheimer Feld 294, 69120 Heidelberg, Germany
E-mail address: marco.bonacini@uni-heidelberg.de

(Hans Knüpfer) Institut für Angewandte Mathematik and Interdisciplinary Center for Scientific Computing (IWR), Universität Heidelberg, 69120 Heidelberg, Germany
E-mail address: knuepfer@uni-heidelberg.de